Functions And Continuity

A **function** from $\mathbb{R}^n$ to $\mathbb{R}^m$ (sometimes called a (vector valued) function of n variables) is a rule which associates to each point in $\mathbb{R}^n$ some point in $\mathbb{R}^m$; the point a function f associates to x is denoted $f(x)$. We write $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (read "f takes $\mathbb{R}^n$ into $\mathbb{R}^m$" or "f, taking $\mathbb{R}^n$ into $\mathbb{R}^m$," depending on context) to indicate that $f(x) \in \mathbb{R}^m$ is defined for $x \in \mathbb{R}^n$. The notation $f : A \longrightarrow \mathbb{R}^m$ indicates that $f(x)$ is defined only for x in the set A, which is called the **domain** of f. If $B \subset A$, we define $f(B)$ as the set of all $f(x)$ for $x \in B$, and if $C \subset \mathbb{R}^m$ we define $f^{-1}(C) = \{x \in A : f(x) \in C\}$. The notation $f : A \longrightarrow B$ indicates that $f(A) \subset B$.

A convenient representation of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ may be obtained by drawing a picture of its graph, the set of all 3-tuples of the form $(x, y, f(x, y))$, which is actually a figure in 3-space (see, e.g., Figures 2-1 and 2-2 of Chapter 2).

If $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$, the functions $f + g$, $f - g$, $f \cdot g$, and f/g are defined precisely as in the one-variable case. If $f : A \longrightarrow \mathbb{R}^m$ and $g : B \longrightarrow \mathbb{R}^p$, where $B \subset \mathbb{R}^m$, then the **composition** $g \circ f$ is defined by $g \circ f(x) = g(f(x))$; the domain of $g \circ f$ is $A \cap f^{-1}(B)$. If $f : A \longrightarrow \mathbb{R}^m$ is 1-1, that is, if $f(x) \neq f(y)$ when $x \neq y$, we define $f^{-1} : f(A) \longrightarrow \mathbb{R}^n$ by the requirement that $f^{-1}(z)$ is the unique $x \in A$ with $f(x) = z$.

A function $f : A \longrightarrow \mathbb{R}^m$ determines m **component functions** $f^1, ..., f^m : A \longrightarrow \mathbb{R}$ by $f(x) = (f^1(x), ..., f^m(x))$. If conversely, m functions $g_1, ..., g_m : A \longrightarrow \mathbb{R}$ are given, there is a unique function $f : A \longrightarrow \mathbb{R}^m$ such that $f^i = g_i$, namely $f(x) = (g_1(x), ..., g_m(x))$. This function f will be denoted $(g_1, ..., g_m)$, so that we always have $f = (f^1, ..., f^m)$. If $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity function, $\pi(x) = x$, then $\pi^i(x) = x^i$; the function π^i is called the ith **projection function**.

The notation $\lim_{x \to a} f(x) = b$ means, as in the one-variable case, that we can get $f(x)$ as close to b as desired, by choosing x sufficiently close to, but not equal to, a. In mathematical terms this means that for every number $\epsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - b| < \epsilon$ for all x in the domain of f which satisfy $0 < |x - a| < \delta$. A function $f : A \longrightarrow \mathbb{R}^m$ is called **continuous** at $a \in A$ if $\lim_{x \to a} f(x) = f(a)$, and f is simply called continuous if it is continuous at each $a \in A$. One of the pleasant surprises about the concept of continuity is that it can be defined without using limits. It follows from the next theorem that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if $f^{-1}(U)$ is open whenever $U \subset \mathbb{R}^m$ is open; if the domain of f is not all of $\mathbb{R}^n$, a slightly more complicated condition is needed.

1-8 Theorem. If $A \subset \mathbb{R}^n$, a function $f : A \longrightarrow \mathbb{R}^m$ is continuous if and only if for every open set $U \subset \mathbb{R}^m$ there is some open set $V \subset \mathbb{R}^n$ such that $f^{-1}(U) = V \cap A$.

Proof. Suppose f is continuous. If $a \in f^{-1}(U)$, then $f(a) \in U$. Since U is open, there is an open rectangle B with $f(a) \in B \subset U$. Since f is continuous at a, we can ensure that $f(x) \in B$, provided we choose x in some sufficiently small rectangle C containing a. Do this for each $a \in f^{-1}(U)$ and let V be the union of all such C. Clearly $f^{-1}(U) = V \cap A$. The converse is similar and is left to the reader. □

Let $a \in A$ and $\epsilon > 0$. Let U be an open rectangle containing a, and within the sphere of radius ϵ centered at $f(a)$. By hypothesis there is some open set $V \subset \mathbb{R}^n$ such that $f^{-1}(U) = V \cap A$. In words, any $x \in V$ that f can act on will get mapped to U. Because V is open and $a \in V$, there is an open rectangle $B \subset V$ that contains a. If δ is the shortest of all the distances from a to the faces of B, then $|x - a| < \delta$ ensures $x \in B$, which ensures that $f(x) \in U$, which ensures that $f(x)$ is within ϵ of $f(a)$. □

The following consequence of Theorem 1-8 is of great importance.

1-9 Theorem. If $f : A \longrightarrow \mathbb{R}^m$ is continuous, where $A \subset \mathbb{R}^n$, and A is compact, then $f(A) \subset \mathbb{R}^m$ is compact.

Proof. Let O be an open cover of $f(A)$. For each open set U in O there is an open set V_U such that $f^{-1}(U) = V_U \cap A$. The collection of all V_U is an open cover of A. Since A is compact, a finite number $V_{U_1}, \ldots, V_{U_n}$ cover A. Then $U_1, \ldots, U_n$ cover $f(A)$. □

If $f : A \longrightarrow \mathbb{R}$ is bounded, the extent to which f fails to be continuous at $a \in A$ can be measured in a precise way. For $\delta > 0$ let

$$M(a, f, \delta) = \sup\{f(x) : x \in A \text{ and } |x - a| < \delta\},$$
$$m(a, f, \delta) = \inf\{f(x) : x \in A \text{ and } |x - a| < \delta\}.$$

The **oscillation** $o(f, a)$ of f at a is defined by

$$o(f, a) = \lim_{\delta \to 0} \left(M(a, f, \delta) - m(a, f, \delta) \right).$$

This limit always exists, since $M(a, f, \delta) - m(a, f, \delta)$ decreases as δ decreases. There are two important facts about $o(f, a)$.

1-10 Theorem. The bounded function f is continuous at a if and only if $o(f,a) = 0$.

Proof. Let f be continuous at a. For every number $\epsilon > 0$ we can choose a number $\delta > 0$ so that $|f(x) - f(a)| < \epsilon$ for all $x \in A$ with $|x - a| < \delta$; thus $M(a,f,\delta) - m(a,f,\delta) \leq 2\epsilon$. Since this is true for every ϵ, we have $o(f,a) = 0$. The converse is similar and is left to the reader. □

If $w \geq 0$ and $w < \epsilon$ for all $\epsilon > 0$, then it must be that $w = 0$. For if $w \neq 0$, then $w < \epsilon$ is violated.

Let $\epsilon > 0$. If $o(f,a) = \lim_{\delta \to 0}(M(a,f,\delta) - m(a,f,\delta)) = 0$, then there is a Δ such that for any $\delta < \Delta$ we have

$$M(a,f,\delta) - m(a,f,\delta) < \epsilon.$$

That is, if $|x - a| < \delta$, then the difference between the least upper bound and the greatest lower bound on $f(x)$ is less than ϵ. The difference between an upper bound and a lower bound of a set is greater than the difference between any two elements of the set, and so we are done: having x within δ of a causes $|f(x) - f(a)|$ to be less than ϵ. □

1-11 Theorem. Let $A \subset \mathbb{R}^n$ be closed. If $f : A \longrightarrow \mathbb{R}$ is any bounded function, and $\epsilon > 0$, then $\{x \in A : o(f,x) \geq \epsilon\}$ is closed.

Proof. Let $B = \{x \in A : o(f,x) \geq \epsilon\}$. We wish to show that $\mathbb{R}^n - B$ is open. If $x \in \mathbb{R}^n - B$, then either $x \notin A$ or else $x \in A$ and $o(f,x) < \epsilon$. In the first case, since A is closed, there is an open rectangle C containing x such that $C \subset \mathbb{R}^n - A \subset \mathbb{R}^n - B$. In the second case there is a $\delta > 0$ such that $M(x,f,\delta) - m(x,f,\delta) < \epsilon$. Let C be an open rectangle containing x such that $|x - y| < \delta$ for all $y \in C$. Then if $y \in C$ there is a δ_1 such that $|x - z| < \delta$ for all z satisfying $|z - y| < \delta_1$. Thus $M(y,f,\delta_1) - m(y,f,\delta_1) < \epsilon$, and consequently $o(y,f) < \epsilon$. Therefore $C \subset \mathbb{R}^n - B$. □

I had to read this several times. Consider a particular y that is within δ of x. For instance, suppose $|x - y| = 0.95 \cdot \delta$. If you stay close enough to this y, your distance to x will still be less than δ.

Problems.

1-23. If $f : A \longrightarrow \mathbb{R}^m$ and $a \in A$, show that $\lim_{x \to a} f(x) = b$ if and only if $\lim_{x \to a} f^i(x) = b^i$ for $i = 1, ..., m$.

From Problem 1-1 we know that $|f(x) - b| \leq \sum_{i=1}^m |f^i(x) - b^i|$. If $\lim_{x \to a} f^i(x) = b^i$, then given $\epsilon > 0$, we can find δ_i such that $|x - a| < \delta_i$ causes $|f^i(x) - b^i| < \epsilon/m$. Making $|x - a|$ less than the smallest δ_i causes $\sum_{i=1}^m |f^i(x) - b^i| < \epsilon$, and therefore $|f(x) - b| < \epsilon$.

In the other direction, note that $u_i^2 \leq u_1^2 + ... + u_m^2$, and so $|u_i| \leq |u|$. If $\lim_{x \to a} f(x) = b$, we can make $|f(x) - b|$ less than some $\epsilon > 0$ by keeping $|x - a|$ less than some δ. Because $|u_i| \leq |u|$, we also have $|f^i(x) - b^i| < \epsilon$.

1-24. Prove that $f : A \longrightarrow \mathbb{R}^m$ is continuous at a if and only if each f^i is.

This follows from the previous problem, replacing b with $f(a)$.

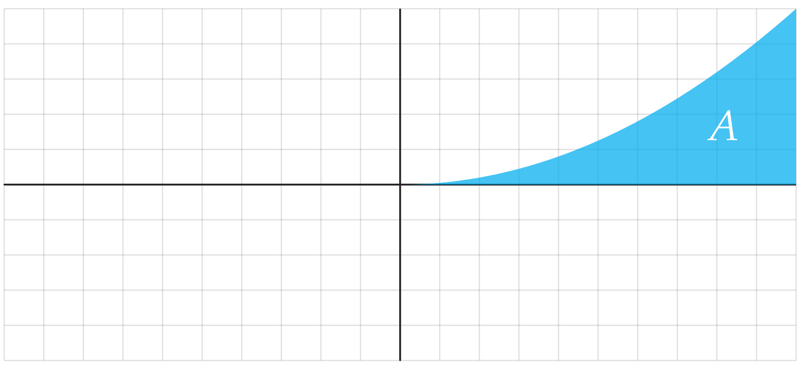
1-25. Prove that a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous. Hint: use Problem 1-10.

From Problem 1-10 we know that $|T(h)| < M|h|$ for some number M. To show continuity at $a \in \mathbb{R}^n$, we need $|T(x) - T(a)|$ to be controlled by $|x - a|$. If $|x - a| < \delta$, then

$$|T(x) - T(a)| = |T(x - a)| < M|x - a| < M\delta.$$

Thus we can make $|T(x) - T(a)| < \epsilon$ by choosing $\delta = \epsilon/M$.

1-26. Let $A = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } 0 < y < x^2\}$.



(a) Show that every straight line through $(0, 0)$ contains an interval around $(0, 0)$ which is in $\mathbb{R}^2 - A$.

The only lines through $(0, 0)$ which intersect A are those given by $y = \alpha x$ for $\alpha > 0$, and these intersect A whenever $x > \alpha$. It follows that on these lines, the interval with $x \in (-\alpha, \alpha)$ is in $\mathbb{R}^2 - A$. All other lines through $(0, 0)$ have no intersection with A, and so any interval on these lines containing $(0, 0)$ is in $\mathbb{R}^2 - A$.

(b) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x) = 0$ if $x \notin A$ and $f(x) = 1$ if $x \in A$. For $h \in \mathbb{R}^2$ define $g_h : \mathbb{R} \longrightarrow \mathbb{R}$ by $g_h(t) = f(th)$. Show that each g_h is continuous at 0, but f is not continuous at $(0, 0)$.

The function $g_h(t)$ radially scales h by the multiplier t, and then samples f at the new point. For small enough t the scaled point th is within the interval established in (a). Therefore, given h, there is an $\epsilon > 0$ such that $g_h(t) = g_h(0) = 0$ for $|t| < \epsilon$. Thus $g_h(t)$ is continuous at 0. In contrast, $f((0, 0)) = 0$ and $f((\epsilon, \epsilon^2/2)) = 1$ for any $\epsilon > 0$, and so f is not continuous at $(0, 0)$.

1-27. Prove that $\{x \in \mathbb{R}^n : |x - a| < r\}$ is open by considering the function $f\colon\mathbb{R}^n \longrightarrow \mathbb{R}$ with $f(x) = |x - a|$.

The function f returns the distance to a given point a. Intuitively, as two points get closer together, so do the distances from these points to a third point a. We formalize this by recalling Problem 1-4 where we showed that $||u| - |v|| \leq |u - v|$. With $u = x - a$ and $v = y - a$, this becomes

$$|f(x) - f(y)| \leq |x - y|.$$

Thus we can ensure $|f(x) - f(y)| < \epsilon$ by picking $|x - y| < \epsilon$. It follows that f is continuous.

If U is the open set (q, r) where q is any negative number, then $f^{-1}(U)$ is the set $\{x \in \mathbb{R}^n : |x - a| < r\}$. Because f is continuous, it follows from Theorem 1-8 that this set is open.

Note that f only maps to the non-negative part of $\mathbb{R}$. No matter! Theorem 1-8 works even though f never reaches parts of U.

1-28. If $A \subset \mathbb{R}^n$ is not closed, show that there is a continuous function $f : A \longrightarrow \mathbb{R}$ which is unbounded. Hint: If $x \in \mathbb{R}^n - A$ but $x \notin$ interior$(\mathbb{R}^n - A)$, let $f(y) = 1/|y - x|$.

If $A \subset \mathbb{R}^n$ is not closed, then $\mathbb{R}^n - A$ is not open, and there exists a point $x \notin A$ with the property that every open rectangle containing x contains a point $y \in A$.

My own difficulty now is being sure that Spivak's $f(y)$ is actually continuous. Certainly as a mapping from $\mathbb{R}^n$ to $\mathbb{R}$, the function $f(y)$ is continuous except at x. But what does continuity mean if the domain of f is some subset of $\mathbb{R}^n$ with a bizarre structure? For instance what if A consists of isolated points? Are assertions of continuity vacuously true? If we claim some property for all y within ϵ of x, but there are no such y, then our claim can't be false because of an absence of counter examples.

This is a preview of coming attractions for me, as the next book on my reading list is Topology by James Munkres.

My concern arises because the definition of x allows A to consist of a sequence of discrete y values. Given $y_k \in A$, we can build an open rectangle about x, all points of which are closer to x than say $|y_k|/2$. This new open rectangle is guaranteed to contain some $y_{k+1} \in A$, and so on. As an example $A \subset \mathbb{R}$ could consist of the points $1/k$ for all integers k.

Let us agree that if $f : A \longrightarrow B$ is continuous and $U \subset A$, then f restricted to U is also continuous. With this agreement we can move forward. As $y \to x$, Spivak's continuous function gets arbitrarily big, so we are done.

1-29. If A is compact, prove that every continuous function $f : A \to \mathbb{R}$ takes on a maximum and a minimum value.

From Theorem 1-9 we know that $f(A)$ is compact, and from Problem 1-20 we know that $f(A) \subset \mathbb{R}$ is closed and bounded. Then from real analysis, we know that a closed and bounded subset of $\mathbb{R}$ contains maximum and minimum elements.

This last fact is easy to prove. Any bounded subset $U \subset \mathbb{R}$ has a least upper bound b, and b is the maximum element of U if $b \in U$. To show that $b \in U$, we construct a sequence of increasing $x_i \in U$ that approach b. We can do this because b is the *least* upper bound of U. Given any $\epsilon > 0$, we can find an x_i within ϵ of b because otherwise $b - \epsilon$ would be an upper bound of U, contrary to hypothesis. Because U is closed, it contains the limit of any sequence of its elements, and so it contains b. A similar argument applies to the minimum element.

1-30. Let $f : [a, b] \longrightarrow \mathbb{R}$ be an increasing function. If $x_1, ..., x_n \in [a, b]$ are distinct, show that

$$\sum_{i=1}^{n} o(f, x_i) < f(b) - f(a).$$

Suppose f is defined on the closed interval $[u, v]$, and $x_i \in (u, v)$. We can find $\delta > 0$ such that $x_i - \delta$ and $x_i + \delta$ are in (u, v). Then, because f is increasing, $o(f, x_i) \leq f(x_i + \delta) - f(x_i - \delta) < f(v) - f(u)$.

Suppose our interval is $[a, v]$ and we care about $o(f, a)$. f isn't defined for $x < a$, and so $o(f, a)$ involves intervals of the form $(a, a + \delta)$. Pick one of these δ and note that because f is increasing, $o(f, a) \leq f(a + \delta) - f(a) < f(v) - f(a)$. Similarly, if our interval is $[u, b]$, then $o(f, b) < f(b) - f(u)$.

Order and relabel the x_i's from the problem statement so that

$$a \leq x_1 < x_2 < ... < x_n \leq b.$$

Let $s_0 = a$, let s_i be some point strictly between x_i and x_{i+1}, let $s_n = b$, and note that $x_i \in [s_{i-1}, s_i]$. From our work above, we know that

$$o(f, x_i) < f(s_i) - f(s_{i-1}).$$

Summing these terms for $i = 1$ to n, we find that everything cancels except $f(s_0 = a)$ and $f(s_n = b)$, leaving us with the desired result.