Subsets of Euclidean Space

The closed interval $[a, b]$ has a natural analogue in $\mathbb{R}^2$. This is the **closed rectangle** $[a, b] \times [c, d]$, defined as the collection of all pairs (x, y) with $x \in [a, b]$ and $y \in [c, d]$. More generally, if $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$, then $A \times B \subset \mathbb{R}^{m+n}$ is defined as the set of all $(x, y) \in \mathbb{R}^{m+n}$ with $x \in A$ and $y \in B$. In particular, $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. If $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$, and $C \subset \mathbb{R}^p$, then $(A \times B) \times C = A \times (B \times C)$, and both of these are denoted simply $A \times B \times C$; this convention is extended to the product of any number of sets. The set $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ is called a **closed rectangle** in $\mathbb{R}^n$, while the set $(a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n$ is called an **open rectangle**. More generally a set $U \subset \mathbb{R}^n$ is called **open** (Figure 1-1) if for each $x \in U$ there is an open rectangle A such that $x \in A \subset U$.

The end construct here seems to be an ordered list of arbitrary length. Building this out of the idea of an ordered pair is awkward, because really, $(A \times B) \times C$ and $A \times (B \times C)$ are not the same. A declaration that they are should be proceeded by "Let us agree that" or something similar.

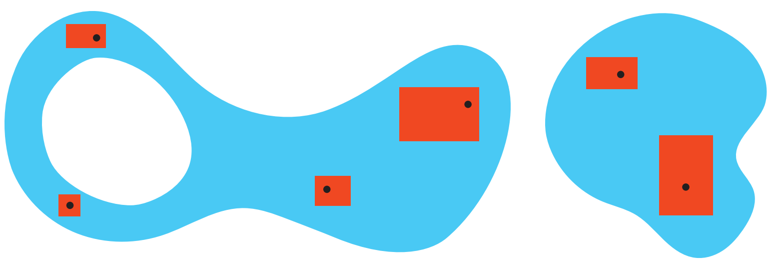


Figure 1-1

A subset C of $\mathbb{R}^n$ is **closed** if $\mathbb{R}^n - C$ is open. For example, if C contains only finitely many points, then C is closed. The reader should supply the proof that a closed rectangle in $\mathbb{R}^n$ is indeed a closed set.

Pick any point x that is not in the given closed rectangle A. The ith component x_i of x is not in the ith closed interval $[a_i, b_i]$ used to define A. It follows that x_i is contained in an open interval (α_i, β_i) which does not overlap $[a_i, b_i]$. The product of these open intervals is an open rectangle containing x which does not overlap A. It follows that $\mathbb{R}^n - A$ is an open set, i.e., that A is a closed set.

If $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, then one of three possibilities must hold (Figure 1-2):

1. There is an open rectangle B such that $x \in B \subset A$.
2. There is an open rectangle B such that $x \in B \subset \mathbb{R}^n - A$.
3. If B is any open rectangle with $x \in B$, then B contains points of both A and $\mathbb{R}^n - A$.

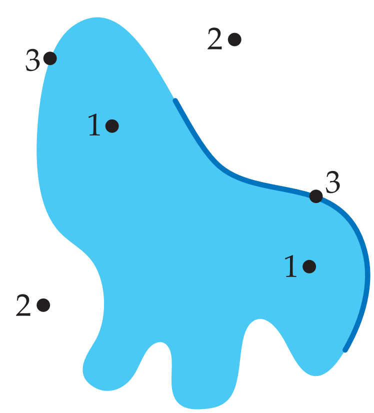


Figure 1-2

Those points satisfying (1) constitute the **interior** of A, those satisfying (2) the **exterior** of A, and those satisfying (3) the **boundary** of A. Problems 1-16 to 1-18 show that these terms may sometimes have unexpected meanings.

It is not hard to see that the interior of any set A is open, and the same is true for the exterior of A, which is, in fact, the interior of $\mathbb{R}^n - A$. Thus (Problem 1-14) their union is open, and what remains, the boundary, must be closed.

A collection O of open sets is an **open cover** of A (or, briefly, **covers** A) if every point $x \in A$ is in some open set in the collection O. For example, if O is the collection of all open intervals $(a, a+1)$ for $a \in \mathbb{R}$, then O is a cover of $\mathbb{R}$. Clearly no finite number of the open sets in O will cover $\mathbb{R}$ or, for that matter, any unbounded subset of $\mathbb{R}$. A similar situation can also occur for bounded sets. If O is the collection of all open intervals $(1/n, 1-1/n)$ for all integers $n > 1$, then O is an open cover of $(0,1)$, but again no finite collection of sets in O will cover $(0,1)$. Although this phenomenon may not appear particularly scandalous, sets for which this state of affairs cannot occur are of such importance that they have received a special designation: a set A is called **compact** if every open cover O contains a finite subcollection of open sets which also covers A.

A set with only finitely many points is obviously compact and so is the infinite set A which contains 0 and the numbers $1/n$ for all integers n (reason: if O is a cover, then $0 \in U$ for some open set U in O; there are only finitely many other points of A not in U, each requiring at most one more open set).

Recognizing compact sets is greatly simplified by the following results, of which only the first has any depth (i.e., uses any facts about the real numbers).

1-3 Theorem (Heine-Borel). The closed interval $[a, b]$ is compact.

A fundamental property of $\mathbb{R}$ is that any subset that is bounded above has a least upper bound. It helps me to think about sets that *lack* this property, such as the rational numbers. If A consists of the rationals that are less than π, then A has a rational upper bound (e.g., 4), however A has no rational *least* upper bound. (Because given any rational upper bound a, it's always possible to find a smaller rational upper bound b.)

Proof. If O is an open cover of $[a, b]$, let

$$A = \{x : a \le x \le b \text{ and } [a, x] \text{ is covered by some finite number of open sets in } O\}.$$

Note that $a \in A$ and that A is clearly bounded above (by b). We would like to show that $b \in A$. This is done by proving two things about $\alpha =$ least upper bound of A; namely, (1) $\alpha \in A$ and (2) $b = \alpha$.

Since O is a cover, $\alpha \in U$ for some U in O. Then all points in some interval to the left of α are also in U (see Figure 1-3). Since α is the least upper bound of A, there is an x in this interval such that $x \in A$.

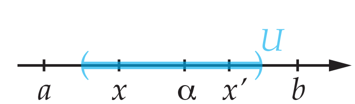


Figure 1-3

Thus $[a, x]$ is covered by some finite number of open sets of O, while $[x, \alpha]$ is covered by the single set U. Hence $[a, \alpha]$ is covered by a finite number of open sets of O, and $\alpha \in A$. This proves (1).

To prove that (2) is true, suppose instead that $\alpha < b$. Then there is a point x' between α and b such that $[\alpha, x'] \subset U$. Since $\alpha \in A$, the interval $[a, \alpha]$ is covered by finitely many open sets of O, while $[\alpha, x']$ is covered by U. Hence $x' \in A$, contradicting the fact that α is an upper bound of A. □

If $B \subset \mathbb{R}^m$ is compact and $x \in \mathbb{R}^n$, it is easy to see that $\{x\} \times B \subset \mathbb{R}^{n+m}$ is compact[1]. However, a much stronger assertion can be made.

[1] If U is a set in $\mathbb{R}^n \times \mathbb{R}^m$, and $x \in \mathbb{R}^n$, let $U_x = \{y \in \mathbb{R}^m : (x, y) \in U\}$. Call U_x the *section* of U by x. Let $y \in U_x$. If U is open, then an open rectangle in U contains (x, y). The section of this rectangle by x is an open rectangle in U_x that contains y. Thus U_x is open.

Let P be an open cover of $\{x\} \times B$, and let Q consist of the sections of the sets in P by x. These sections are open, and they cover B. Because B is compact, it is covered by a finite number of these sections. The corresponding (finitely many) sets in P cover $\{x\} \times B$. □

1-4 Theorem. If B is compact and O is an open cover of $\{x\} \times B$, then there is an open set $U \subset \mathbb{R}^n$ containing x such that $U \times B$ is covered by a finite number of sets in O.

Proof. Since $\{x\} \times B$ is compact, we can assume at the outset that O is finite, and we need only find the open set U such that $U \times B$ is covered by O.

For each $y \in B$ the point (x, y) is in some open set W in O. Since W is open, we have $(x, y) \in U_y \times V_y \subset W$ for some open rectangle $U_y \times V_y$. The sets V_y cover the compact set B, so a finite number $V_{y1}, \ldots, V_{yk}$ also cover B. Let $U = U_{y1} \cap \ldots \cap U_{yk}$. Then if $(x', y') \in U \times B$, we have $y' \in V_{yi}$ for some i (Figure 1-4), and certainly $x' \in U_{yi}$. Hence $(x', y') \in U_{yi} \times V_{yi}$, which is contained in some W in O. □

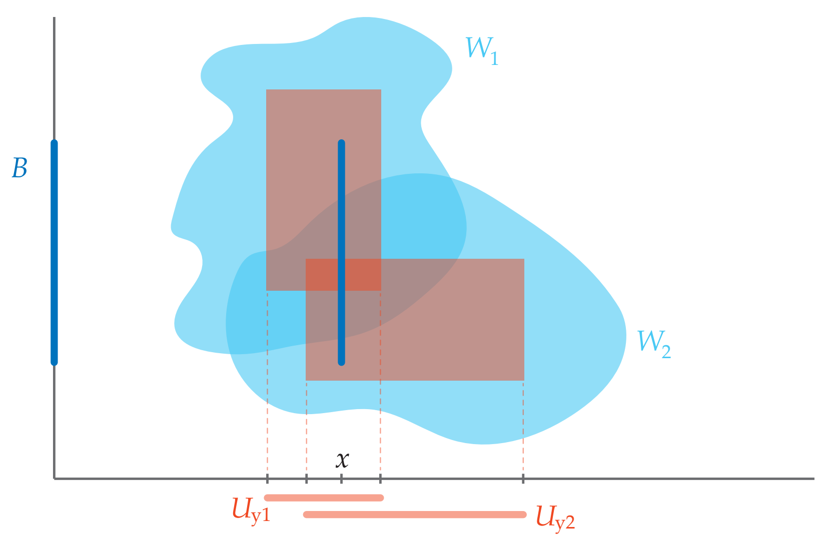


Figure 1-4

1-5 Corollary. If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are compact, then $A \times B \subset \mathbb{R}^{n+m}$ is compact.

Proof. If O is an open cover of $A \times B$, then O covers $\{x\} \times B$ for each $x \in A$. By Theorem 1-4 there is an open set U_x containing x such that $U_x \times B$ is covered by finitely many sets in O. Since A is compact, a finite number $U_{x1}, ..., U_{xn}$ of the U_x cover A. Since finitely many sets in O cover each $U_{xi} \times B$, finitely many cover all of $A \times B$. □

1-6 Corollary. $A_1 \times \cdots \times A_k$ is compact if each A_i is. In particular, a closed rectangle in $\mathbb{R}^k$ is compact.

1-7 Corollary. A closed bounded subset of $\mathbb{R}^n$ is compact. (The converse is also true (Problem 1-20).)

Proof. If $A \subset \mathbb{R}^n$ is closed and bounded, then $A \subset B$ for some closed rectangle B. If O is an open cover of A, then O together with $\mathbb{R}^n - A$ is an open cover of B. Hence a finite number $U_1, ..., U_n$ of sets in O, together with $\mathbb{R}^n - A$ perhaps, cover B. Then $U_1, ..., U_n$ cover A. □

Problems.

1-14.* Prove that the union of any (even infinite) number of open sets is open. Prove that the intersection of two (and hence of finitely many) open sets is open. Give a counterexample for infinitely many open sets.

Let $\mathcal{A}$ be a (possibly infinite) collection of open sets. Let x be in the union U of these sets, (i.e., $x \in A$ for at least one $A \in \mathcal{A}$). Because A is open, it contains an open rectangle that contains x. Everything in A is part of the union, including this open rectangle. Thus for any $x \in U$, we have an open rectangle in U that contains x. It follows that U is open.

Let x be in the intersection of two open sets A and B. Because A is open, x belongs to an open rectangle in A, formed by the product $\alpha_1 \times \alpha_2 \times ... \times \alpha_n$, where each α_i is an open interval on $\mathbb{R}$. Likewise for B, where the open rectangle is given by $\beta_1 \times \beta_2 \times ... \times \beta_n$. The ith intervals α_i and β_i overlap because they both contain the ith component of x. The intersections $\alpha_i \cap \beta_i$ give us a new open rectangle containing x which is in both A and B. It follows that $A \cap B$ is open.

Consider the open sets $(-1/n, 1/n)$ for $n = 1, 2, 3, ...$ The intersection of this infinite collection of sets is the single point at zero. Any set consisting of a single point is not open.

1-15. Prove that $\{x \in \mathbb{R}^n : |x-a| < r\}$ is open (see also Problem 1-27).

Let B (for ball) denote the specified set, and let $x \in B$. Any y that is close enough to x will be within B. Specifically, if $|y-x| < \epsilon$, with $\epsilon = r - |a-x|$, then

$$\begin{aligned} |y-a| &= |y-x-(a-x)| \\ &\leq |y-x| + |a-x| \\ &< r. \end{aligned}$$

As a result, if R is an open rectangle centered at x, all of whose points are closer to x than ϵ, then $R \subset B$. We build such an R as a hypercube centered at x and with sides of length $2l$.

$$R = (x_1 - l, x_1 + l) \times (x_2 - l, x_2 + l) \times \ldots \times (x_n - l, x_n + l)$$

The distance between x and any $u \in R$ is less than $\sqrt{nl^2}$. Therefore, setting $l = \epsilon/\sqrt{n}$ causes every point in R to be closer to x than ϵ, which causes $R \subset B$. Remember that R contains x, which was chosen arbitrarily within B, and so B is open.

1-16. Find the interior, exterior, and boundary of the sets

$$\begin{aligned} &\{x \in \mathbb{R}^n : |x| \leq 1\} \\ &\{x \in \mathbb{R}^n : |x| = 1\} \\ &\{x \in \mathbb{R}^n : \text{each } x^i \text{ is rational}\}. \end{aligned}$$

Let $A = \{x \in \mathbb{R}^n : |x| \leq 1\}$.

i. If $|x| < 1$, then $x \in B \subset A$, where B is from the previous problem. We showed that B is open, and so any $x \in B$ is surrounded by an open rectangle entirely within B. This open rectangle is also entirely within A. It follows that x is in the interior of A.

ii. Let $|x| = 1$, and let R be an open rectangle containing x. If ϵ is the shortest distance from x to a face of R, then any y which satisfies $|x-y| < \epsilon$ will also be in R. $y = x + \frac{\epsilon}{2|x|}x$ satisfies this condition, and is not in A. At the same time, $x \in R$ and is in A. It follows that x is part of the boundary of A.

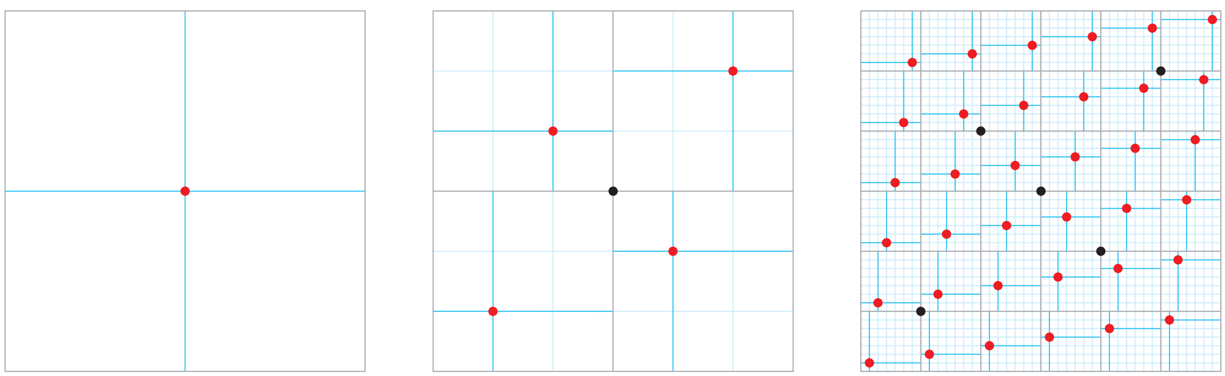
iii. Let $|x| > 1$. From Problem 1-15, we know we can build an open rectangle containing x such that the maximum distance from x to any point in the rectangle is less than some $\epsilon > 0$. Picking $\epsilon = |x| - 1$ ensures that no point in the rectangle is in A. It follows that x is part of the exterior of A.

Let $A = \{x \in \mathbb{R}^n : |x| = 1\}$.
From ii. above, we can build an open rectangle around any $x \in A$ which contains points that are not in A. It follows that every $x \in A$ is a boundary point. From iii. above, we know that any $|x| > 1$ is part of the exterior of A. By a similar argument, we know that any $|x| < 1$ is also part of the exterior of A. It follows that A has no interior.

Let $A = \{x \in \mathbb{R}^n : \text{each } x_i \text{ is rational}\}$.
Let $x \in \mathbb{R}^n$, and let R be an open rectangle containing x. The ith coordinate of x belongs to some open interval (a_i, b_i) that defines R in the ith dimension. We know from real analysis that there are rational and irrational numbers on any open interval, so let u_i and v_i be rational and irrational respectively on (a_i, b_i). The corresponding vectors $u \in A$ and $v \notin A$ are both in R. It follows that x is part of the boundary of A. But x was arbitrary, and so the boundary of A is all of $\mathbb{R}^n$. It follows as a corollary that A has no interior and no exterior.

1-17. Construct a set $A \subset [0,1] \times [0,1]$ such that A contains at most one point on each horizontal and each vertical line but boundary $A = [0,1] \times [0,1]$. Hint: It suffices to ensure that A contains points in each quarter of the square $[0,1] \times [0,1]$ and also in each sixteenth, etc.

We build A as the infinite union $A_0 \cup A_1 \cup A_2 \cup \ldots$



A_0 $\qquad$ $A_0 \cup A_1$ $\qquad$ $A_0 \cup A_1 \cup A_2$

A_0 is the set containing the single point $(\frac{1}{2}, \frac{1}{2})$.

To build A_k, draw vertical and horizontal lines through all the points in $A_0 \cup \ldots \cup A_{k-1}$, creating an $N \times N$ checkerboard of squares. Then draw a single dot in each square as follows.

Let (m, n) identify the square m from the left, and n from the bottom, with each index ranging from 1 to N. Create an $(N+1) \times (N+1)$ grid *within the small square at* (m, n) by drawing N evenly spaced horizontal lines, and N evenly spaced vertical lines. Place a single point at the intersection of horizontal line m and vertical line n.

If R is any open rectangle around any point in $[0,1] \times [0,1]$, then for high enough k our construction will result in a grid with small squares some of which are contained in R. Our construction places a point in each of these squares, and so R contains a point from A. The coordinates of the points in A are rational, and so by the same argument used in the previous problem, we know that R contains points that are not in A. It follows that every point in $[0,1] \times [0,1]$ is a boundary point. Finally, by construction we know that A contains at most one point on each horizontal and vertical line.

1-18. If $A \subset [0,1]$ is the union of open intervals (a_i, b_i) such that each rational number in $(0,1)$ is contained in some (a_i, b_i), show that boundary $A = [0,1] - A$.

The set A depends on the open intervals (a_i, b_i) that are assigned to each rational number on $(0,1)$. For instance

Each $(a_i, b_i) \subset [0,1]$ because otherwise we fail to have $A \subset [0,1]$.

- If each rational number is assigned to $(0,1)$, then $A = (0,1)$.
- Let $\gamma \in (0,1)$ be irrational, (e.g., equal to $\pi/4$). If rational numbers less than γ are assigned to $(0,\gamma)$, and rational numbers greater than γ are assigned to $(\gamma,1)$, then $A = (0,1) - \{\gamma\}$.

Spivak's claim is that $\partial A = [0,1] - A$ in *all* cases, that is, regardless of the assignment of open intervals to rational numbers. To see this, note that:

∂A denotes the boundary of A.

i. If $x \in A$, then x is in the interior of A, because x is contained in an open interval (a_i, b_i), all of whose elements belong to A.

ii. If $x \notin [0,1]$, then x is in the exterior of A, because we can enclose x in an open interval that does not intersect $[0,1]$.

iii. If $x \in [0,1] - A$, we know that there are rationals arbitrarily close to x which belong to A. Any open interval that contains x (which is not in A) will also contain these rationals (which are in A). It follows that $x \in \partial A$.

Any rational belongs to A because it is contained in its associated open interval (a_i, b_i) which belongs to A.

In iii. we establish that $x \in [0,1] - A$ implies $x \in \partial A$. In i. and ii. we establish that $x \notin [0,1] - A$ implies $x \notin \partial A$. It follows that $\partial A = [0,1] - A$ as desired.

1-19.* If A is a closed set that contains every rational number $r \in [0,1]$, show that $[0,1] \subset A$.

If A is a closed set that contains every rational number on $[0,1]$, then the complement of A (denoted $\bar{A}$) is an open set that contains no rational numbers on $[0,1]$.

Suppose $\bar{A}$ includes an irrational $x \in [0,1]$. Because $\bar{A}$ is open, it contains an open interval containing x. But any open interval containing x contains rational numbers on $[0,1]$, and $\bar{A}$ contains none of these. It must be that $\bar{A}$ includes no irrational numbers on $[0,1]$.

We now know that $\bar{A}$ is an open set that contains no numbers on $[0,1]$. It follows that A is a closed set that contains all numbers on $[0,1]$. We write this as $[0,1] \subset A$.

1-20. Prove the converse of Corollary 1-7: A compact subset of $\mathbb{R}^n$ is closed and bounded (see also Problem 1-28).

If a set A is not closed, then the complement of A is not open, and we know that some point x in the complement of A exists with the following property:

Not closed is different than open. Sets that are both closed *and* open are called "clopen".

Every open rectangle containing x also contains some $y \in A$. ($\bigstar$)

Consider the open cover of A consisting of the complements of closed rectangles containing x. Each of these anti-rectangles is open, and together they cover every point in $\mathbb{R}^n$ except for x. Therefore they cover A, because A consists of points in $\mathbb{R}^n$ except for x. Notice that the union of any *finite* subset of these anti-rectangles leaves a gap around x, within which we can place an open rectangle containing x. From ($\bigstar$) we know that this open rectangle contains some $y \in A$, and so our finite subset of anti-rectangles does not cover A. We chose this subset arbitrarily, and so we've established that A is not compact. Showing that A is not closed implies A is not compact is the same as showing that any compact set is closed.

To show that any compact set A is bounded, we begin with an open cover consisting of unit hypercubes centered at every point in $\mathbb{R}^n$. Because A is compact, it is covered by some finite number of these unit hypercubes. A rectangle containing a finite number of unit hypercubes can be constructed by taking the minimum and maximum of their coordinates in each dimension. This rectangle contains A and so A is bounded.

1-21.* (a) If A is closed and $x \notin A$, prove that there is a number $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.

If A is closed and $x \notin A$, then x belongs to an open set (the complement of A). An open rectangle exists that contains x and that is wholly within the complement of A. If d is the shortest distance from a face of this rectangle to x, then $|y - x| \geq d$ for all y outside of the rectangle, in particular for $y \in A$.

(b) If A is closed, B is compact, and $A \cap B = \emptyset$, prove that there is $d > 0$ such that $|y - x| \geq d$ for all $y \in A$ and $x \in B$. *Hint*: For each $b \in B$ find an open set U containing b such that this relation holds for $x \in U \cap B$.

In (a), we were working with a particular x. Here we need the same outcome as (a), but true for all $x \in B$.

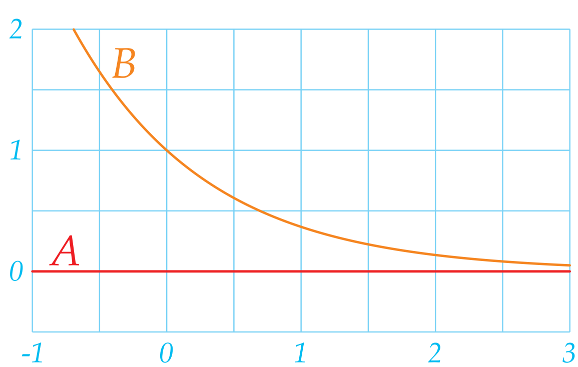
Associate to each $x \in B$ the open sphere of radius d centered at x, where d is provided by (a). See Problem 1-15 for certainty that the sphere is open, and note that (a) ensures that the sphere has no intersection with A. The set of these spheres over all $x \in B$ is an open cover of B, and because B is compact, a finite sub collection of these spheres also covers B. The minimum of the associated d values satisfies the given problem.

(c) Give a counterexample in $\mathbb{R}^2$ if A and B are closed but neither is compact.

A curve is a closed set in $\mathbb{R}^2$. A curve isn't compact if it isn't bounded (i.e., doesn't fit within some rectangle). Consider

$$A = \{(x, y) | y = 0\} \quad \text{and} \quad B = \{(x, y) | y = e^{-x}\}.$$

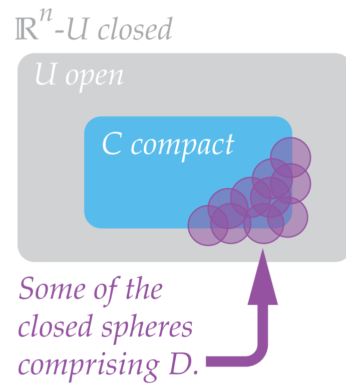
By making x large enough, we can find points $(x, 0) \in A$ and $(x, e^{-x}) \in B$ that are arbitrarily close together.



1-22.* If U is open and $C \subset U$ is compact, show that there is a compact set D such that $C \subset$ interior D and $D \subset U$.

The complement of U is closed and has no intersection with the compact set C. From Problem 1-21, we know some $d > 0$ exists for which $|x - y| \geq d$ for any $x \notin U$ and $y \in C$.

Consider the open cover of C consisting of open spheres of radius $d/2$ centered at points in C. Because C is compact, some finite sub collection of these spheres also covers C. Let D be the (finite) union of these same spheres plus their boundaries. D is the union of finitely many *closed* spheres, and so D is closed. Also, by construction we have $C \subset$ interior D and $D \subset U$.



To show that D is compact, we need to show that D is bounded. Every point in D is within $d/2$ of C, which is compact and therefore bounded, (i.e., contained in a rectangle). Expanding a rectangle bounding C in all dimensions by d results in a new rectangle that bounds D.