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1 Notes on The Partial Fractions Expansion

1.1 Overview

These notes examine the partial fractions expansion in a linear algebra setting. The expansion is justified for quotients of general complex valued polynomials, and then specialized to quotients of polynomials with real coefficients.

1.2 The General Setting

Let $q(s) : \mathbb{C} \longrightarrow \mathbb{C}$ be a monic polynomial of degree n with complex coefficients. The fundamental theorem of algebra guarantees the existance of $l \leq n$ distinct complex roots r_i such that $q(s)$ can be written as

$$q(s) = \prod_{i=1}^{l} (s - r_i)^{n_i} \tag{1}$$

where n_i is the multiplicity of the root r_i, and $\sum n_i = n$.

1.3 Distinct Roots

We first consider the case where $n_i = 1$ for every i. This causes there to be n distinct roots r_i, and correspondingly n distinct terms $(s - r_i)$ in (1). Now consider a polynomial $p(s)$ of degree $m < n$ with complex coefficients. We question whether or not numbers $c_i \in \mathbb{C}$ exist that allow us to represent the quotient $p(s)/q(s)$ as follows

$$\frac{p(s)}{q(s)} = \frac{c_1}{(s - r_1)} + \frac{c_2}{(s - r_2)} + \cdots + \frac{c_n}{(s - r_n)} \tag{2}$$

We multiply each term $c_j/(s - r_j)$ above by the convenient form of unity $b_j(s)/b_j(s)$ with

$$b_j(s) = \prod_{i \in J} (s - r_i) \tag{3}$$

where J is the set of all integers from 1 to n with the exception of j. This causes (2) to take the form

$$\frac{p(s)}{q(s)} = \sum_{j=1}^{n} \frac{c_j b_j(s)}{q(s)} \tag{4}$$

Therefore our question becomes one of whether or not $\{b_j(s)\}$ spans the space of complex polynomials of degree m. This will occur if $\{b_j(s)\}$ is a basis of the space of complex polynomials of degree $n - 1$ (recall that $m < n$). The space of complex polynomials of degree $n - 1$ has dimension n, which is the same as the number of elements in $\{b_j(s)\}$, and so the $b_j(s)$'s comprise a basis if they are linearly independent. Suppose that n numbers $\alpha_j \in \mathbb{C}$ exist so that $\sum \alpha_j b_j(s) = 0$. This sum can be expanded as follows

$$\begin{aligned} \alpha_j b_j(s) = &\alpha_1 (\quad\quad (s - r_2)(s - r_3)(s - r_4) \cdots) \\ &\alpha_2 ((s - r_1) \quad\quad (s - r_3)(s - r_4) \cdots) \\ &\alpha_3 ((s - r_1)(s - r_2) \quad\quad (s - r_4) \cdots) \\ &\alpha_4 ((s - r_1)(s - r_2)(s - r_3) \quad\quad \cdots) \\ &+ \cdots \end{aligned} \tag{5}$$

where a space emphasizes the missing term in each $b_j(s)$. Picking $s = r_1$ causes every row except the first one to be zero (recall that we are considering the case where the roots r_i are distinct, and so $b_1(r_1) \neq 0$). The condition $\sum \alpha_j b_j(s)$ then holds only if $\alpha_1 = 0$. Continuing in this manner, we find that $\alpha_j = 0$ for every j, and so the $b_j(s)$'s are linearly independent as desired. It follows that $\{b_j(s)\}$ is a basis of the space of complex polynomials of degree $n - 1$, and so c_i exist which allow the representation (2).

1.4 Repeated Roots

We now turn our attention to the general case, where n_i are allowed to be greater than 1. As before, we will show that the question of the validity of a representation is equivilant to the question of whether or not some set of polynomials is a basis for all polynomials of a certain degree. A representation of $p(s)/q(s)$ (where $p(s)$ is still some degree $m < n$ polynomial with complex coefficients) of the form (2) is inadequate when the roots of $q(s)$ are repeated. Instead we try the following representation

$$
\begin{aligned}
\frac{p(s)}{q(s)} =& \frac{c_{11}}{(s-r_1)} + \frac{c_{12}}{(s-r_1)^2} + \cdots + \frac{c_{1n_1}}{(s-r_1)^{n_1}} \\
&+ \frac{c_{21}}{(s-r_2)} + \frac{c_{22}}{(s-r_2)^2} + \cdots + \frac{c_{2n_2}}{(s-r_2)^{n_2}} \\
&\vdots \\
&+ \frac{c_{l1}}{(s-r_l)} + \frac{c_{l2}}{(s-r_l)^2} + \cdots + \frac{c_{ln_l}}{(s-r_l)^{n_l}}
\end{aligned} \tag{6}
$$

Following our developement for the simple case, each quotient above is multiplied by the convenient form of unity which causes its denominator to equal $q(s)$. If the numerator polynomials resulting from this multiplication comprise a basis for the space of degree $n-1$ polynomials, then c_{ij} allowing this representation will be sure to exist. The n numerator polynomials are enumerated as $b_k(s)$, where the c_{ij} term in (6) corresponds to $b_{n_1+n_2+\cdots+n_{i-1}+j}(s)$ (with $n_0 = 0$). That is, the k in $b_k(s)$ is 1 when $b_k(s0$ corresponds to the c_{11} term in (6), and then increases as we move to the right through the first row, and from left to right through each of the rows that follow (taken in order from the top of (6) to the bottom). We now consider the implications of the condition $\alpha_j b_j(s) = 0$. Writing out this sum explicitly, we find that

$$
\begin{aligned}
\alpha_j b_j(s) = \;& \alpha_1 \left((s-r_1)^{n_1-1}(s-r_2)^{n_2}(s-r_3)^{n_3}\cdots(s-r_l)^{n_l}\right) \\
&+ \alpha_2 \left((s-r_1)^{n_1-2}(s-r_2)^{n_2}(s-r_3)^{n_3}\cdots(s-r_l)^{n_l}\right) \\
&\vdots \\
&+ \alpha_{n_1} \left((s-r_1)^{0}\quad(s-r_2)^{n_2}(s-r_3)^{n_3}\cdots(s-r_l)^{n_l}\right) \\
&\\
&+ \alpha_{n_1+1} \left((s-r_1)^{n_1}(s-r_2)^{n_2-1}(s-r_3)^{n_3}\cdots(s-r_l)^{n_l}\right) \\
&+ \alpha_{n_1+2} \left((s-r_1)^{n_1}(s-r_2)^{n_2-2}(s-r_3)^{n_3}\cdots(s-r_l)^{n_l}\right) \\
&\vdots \\
&+ \alpha_{n_1+n_2} \left((s-r_1)^{n_1}(s-r_2)^{0}\quad(s-r_3)^{n_3}\cdots(s-r_l)^{n_l}\right) \\
&\vdots \\
&\vdots \\
&+ \alpha_{n_1+\cdots+n_{l-1}+1} \left((s-r_1)^{n_1}(s-r_2)^{n_2}(s-r_3)^{n_3}\cdots(s-r_l)^{n_l-1}\right) \\
&+ \alpha_{n_1+\cdots+n_{l-1}+2} \left((s-r_1)^{n_1}(s-r_2)^{n_2}(s-r_3)^{n_3}\cdots(s-r_l)^{n_l-2}\right) \\
&\vdots \\
&+ \alpha_{n_1+\cdots+n_{l-1}+n_l} \left((s-r_1)^{n_1}(s-r_2)^{n_2}(s-r_3)^{n_3}\cdots(s-r_l)^{0}\quad\right)
\end{aligned} \tag{7}
$$

For clarity we raise certain terms to the power 0; keep in mind that these terms have value 1. Setting $s = r_1$ causes every thing except the n_1 row above to equal zero, and so we see that $\alpha_{n_1} = 0$. Next, note that $(s - r_1)$ can be factored out of every polynomial that remains. Doing this and setting $s = r_1$ a second time causes all rows except the $n_1 - 1$ row to equal zero. If something that is at most an order k polynomial equals zero at $k + 1$ points, then it is the zero function. With this in mind we see that $\alpha_{n_1-1} = 0$. Continuing in this fashion, we find that all the α's are zero. The polynomials $b_j(s)$ are therefore linearly independent, and comprise a basis of the dimension n space of degree $n-1$ complex polynomials. As a result, we can be sure that numbers c_{ij} exist which allow the representation (6).

1.5 Real Coefficients

In the previous section, we showed that a convenient *partial fractions expansion* exists of the quotient $p(s)/q(s)$, where $q(s)$ is a polynomial of degree n, and $p(s)$ is a monic complex polynomial of degree $m < n$. This result continues to hold when the polynomial coefficients are real, however it is inconvenient when expressed as (6), because the coefficients c_{ij} in this representation will generally be complex, even though $q(s)$ and $p(s)$ acting on real arguments result in a quotient $p(s)/q(s)$ that is always real valued. In this section we develop a form of (6) that is suited to the case where $p(s)$ and $q(s)$ have real coefficients, and where the argument s is also real.

If $q(s)$ has real coefficients, then its roots r are real, or occur in complex conjugate pairs. This means that if r is a complex root of $q(s)$ with multiplicity $\tilde{n}$, then its conjugate $\bar{r}$ is also a root of $q(s)$ with multiplicity $\tilde{n}$. Note that because the quotient $Q(s) = p(s)/q(s)$ is real when s is real, the reflection principle guarantees that $Q(\bar{s}) = \overline{Q(s)}$. Focusing on rows i and j of (6), which we suppose are the rows corresponding to r and $\bar{r}$, we find that

$$\begin{aligned}
Q(\bar{s}) =& \frac{c_{i1}}{\bar{s}-r} + \frac{c_{i2}}{(\bar{s}-r)^2} + \cdots + \frac{c_{i\tilde{n}}}{(\bar{s}-r)^{\tilde{n}}} \\
&+ \frac{c_{j1}}{\bar{s}-\bar{r}} + \frac{c_{j2}}{(\bar{s}-\bar{r})^2} + \cdots + \frac{c_{j\tilde{n}}}{(\bar{s}-\bar{r})^{\tilde{n}}} \\
&+ \quad other \quad rows \\
\overline{Q(s)} =& \frac{\bar{c}_{i1}}{\bar{s}-\bar{r}} + \frac{\bar{c}_{i2}}{(\bar{s}-\bar{r})^2} + \cdots + \frac{\bar{c}_{i\tilde{n}}}{(\bar{s}-\bar{r})^{\tilde{n}}} \\
&+ \frac{\bar{c}_{j1}}{\bar{s}-r} + \frac{\bar{c}_{j2}}{(\bar{s}-r)^2} + \cdots + \frac{\bar{c}_{j\tilde{n}}}{(\bar{s}-r)^{\tilde{n}}} \\
&+ \quad other \quad rows
\end{aligned} \tag{8}$$

Matching terms (as we must if s is free to vary), we find that $c_{jk} = \bar{c}_{ik}$. With this in mind, we combine corresponding terms of $Q(s)$ as follows

$$\frac{c_{ik}}{(x-r)^k} + \frac{c_{jk}}{(x-\bar{r})^k} = \frac{c_{ik}\overline{(x-r)^n} + \bar{c}_{ik}(x-r)^n}{(x^2+\alpha x+\beta)^n} = \frac{p_n(x)}{(x^2+\alpha x+\beta)^n} \tag{9}$$

where $\alpha^2 < 4\beta$, and where we have replaced s with the real variable x. We denote the numerator $p_n(x)$ because it is a real n^{th}-degree polynomial in x. Construction (9) allows us to combine the two rows of $Q(x)$ that are due to r into a single row as follows

$$\begin{aligned}
Q(x) =& \frac{p_1(x)}{x^2+\alpha x+\beta} + \frac{p_2(x)}{(x^2+\alpha x+\beta)^2} + \cdots + \frac{p_{\tilde{n}}(x)}{(x^2+\alpha x+\beta)^{\tilde{n}}} \\
&+ \quad other \quad rows
\end{aligned} \tag{10}$$

The terms in this condensed row can be combined to form a single quotient, with $(x^2+\alpha x+\beta)^{\tilde{n}}$ in the denominator, and some degree $\tilde{n}$ polynomial in the numerator. This single quotient can be expressed as

$$\frac{p_{\tilde{n}}(x)}{(x^2+\alpha x+\beta)^{\tilde{n}}} = \frac{a_{i1}x+b_{i1}}{x^2+\alpha x+\beta} + \frac{a_{i2}x+b_{i2}}{(x^2+\alpha x+\beta)^2} + \cdots + \frac{a_{i\tilde{n}}x+b_{i\tilde{n}}}{(x^2+\alpha x+\beta)^{\tilde{n}}} \tag{11}$$

This is easy to see: simply return (11) to its single quotient form by multiplying each term by a convenient form of unity. Then a_{i1} will be the coefficient of a degree $2\tilde{n}$ polynomial, b_{i1} will be the coefficient of a degree $2\tilde{n}-1$ polynomial, and so on, with $a_{i\tilde{n}}$ the coefficient of a degree 1 polynomial (that is of x), and with $b_{i\tilde{n}}$ a constant. Clearly values of these coefficients exist that cause the numerator of the single quotient form of (11) to equal any degree $\tilde{n}$ polynomial, and so (11) is indeed valid. This completes our adaptation of the partial fractions expansion to quotients of real valued polynomials. The adaptation to real coefficients is correct, but not at all elegent. I had hoped to find an expression for the coefficients a_{ij} and b_{ij} in (11) in terms of the real and imaginary parts of the initial expansion coefficients. My first investigations suggest that these expressions are not at all nice, and consist of large sums. A cleaner developement can probably be had by specializing to the reals earlier on.

1.6 Observations

The partial fractions expansion is unique, as is the expression of any vector with respect to a basis of its resident vector space.

1.7 Polynomial Representations

The following is a neat expression for the polynomial that is generated by multiplying n terms of the form $(x + \lambda_i)$ together

$$\sum_{i=1}^{n}(x+\lambda_i) = \prod_{j=0}^{n} x^{n-j}\lambda_{(j)}, \quad \text{where} \quad \lambda_{(j)} = \sum_{\alpha_1=1}^{n}\sum_{\alpha_2=\alpha_1}^{n}\cdots\sum_{\alpha_j=\alpha_{j-1}}^{n}\left(\lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_j}\delta_{\alpha_1\alpha_2\cdots\alpha_k}\right) \tag{12}$$

More explicitly,

$$(x+\lambda_1)(x+\lambda_2)\cdots(x+\lambda_n) = x^n + x^{n-1}\lambda_{(1)} + \cdots + x\lambda_{(n-1)} + \lambda_{(n)} \tag{13}$$

where $\lambda_{(j)}$ is the sum of all unique j-tuple products of λ_i, each of which can include a particular λ_i at most once, and where $\lambda_{(0)} = 1$). As an example,

- for n=1

$$\begin{aligned}(x+\lambda_1) = x& \\ + \lambda_1&\end{aligned}$$

- for n=2

$$\begin{aligned}&(x+\lambda_1)(x+\lambda_2)\\ &\quad = x^2\\ &\quad + x(\lambda_1+\lambda_2)\\ &\quad + \lambda_1\lambda_2\end{aligned}$$

- for n=3

$$\begin{aligned}&(x+\lambda_1)(x+\lambda_2)(x+\lambda_3)\\ &\quad = x^3\\ &\quad + x^2(\lambda_1+\lambda_2+\lambda_3)\\ &\quad + x(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\\ &\quad + \lambda_1\lambda_2\lambda_3\end{aligned}$$

- for n=4

$$\begin{aligned}&(x+\lambda_1)(x+\lambda_2)(x+\lambda_3)(x+\lambda_4)\\ &\quad = x^4\\ &\quad + x^3(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\\ &\quad + x^2(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)\\ &\quad + x(\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4)\\ &\quad + \lambda_1\lambda_2\lambda_3\lambda_4\end{aligned}$$