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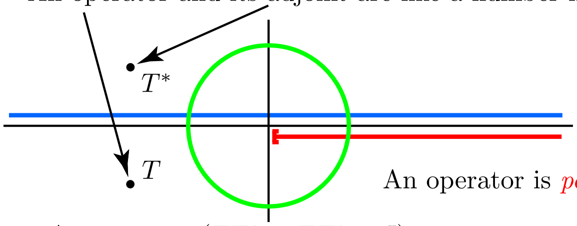
Summary of Operator Facts and Terminology

The following is a summary of the major categories of operators on finite dimensional inner product vector spaces, as presented in the wonderful *Linear Algebra Done Right* by Sheldon Axler.

An operator and its adjoint are like a number in $\mathbb{C}$ and its conjugate.

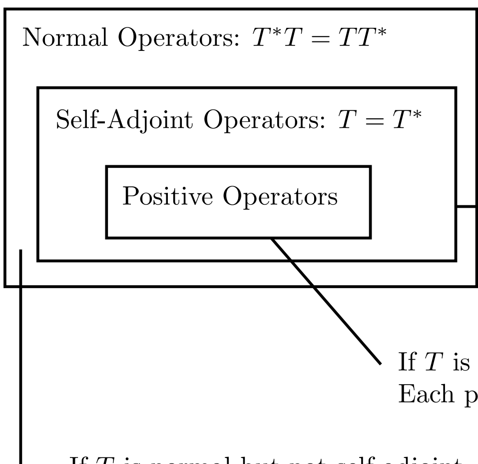


self-adjoint ($T = T^*$) operators have all real eigenvalues.
- these are also called *hermitian* operators.
- if $\mathbb{F} = \mathbb{R}$ then the matrix of the operator wrt any orthonormal basis is *symmetric*.

An operator is *positive* if it is self-adjoint and if $\langle Tv, v \rangle \geq 0 \;\; \forall v \in V$.

- An *isometry* ($TT^* = TT^* = I$) occupies something analogous to the unit circle.
- called *orthogonal* if $\mathbb{F} = \mathbb{R}$.
- called *unitary* if $\mathbb{F} = \mathbb{C}$.

An operator is *normal* iff $TT^* = TT^*$ (analogy with $\mathbb{C}$ ignored because every point in the complex plane would correspond to a normal operator).



The complex spectral theorem applies to all normal operators when $\mathbb{F} = \mathbb{C}$. It guarantees that $\exists$ an orthonormal basis (e_i) s.t. $M(T, (e_i)) = \text{diag}(\lambda_1, \cdots, \lambda_n)$ where $\lambda_i \in \text{Spec}(T)$.

The real spectral theorem applies to all self-adjoint operators when $\mathbb{F} = \mathbb{R}$. It guarantees that $\exists$ an orthonormal basis (e_i) s.t. $M(T, (e_i)) = \text{diag}(\lambda_1, \cdots, \lambda_n)$ where $\lambda_i \in \text{Spec}(T)$.

If T is positive, then each $\lambda_i \in \text{Spec}(T)$ is non-negative.
Each positive operator has a unique positive square root.

If T is normal but not self-adjoint, and if $\mathbb{F} = \mathbb{R}$, then $\exists$ an orthonormal basis (e_i) s.t. $M(T, (e_i))$ is block diagonal, with blocks that are either 1×1 scalars in $\mathbb{R}$, or 2×2, of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $b > 0$, (with at least one of these 2×2 blocks, or T would be self-adjoint).

Isometries:

- $\mathbb{F} = \mathbb{C}$ and S an isometry $\iff \exists$ an orthonormal basis (e_i) of eigenvectors of S, each with corresponding eigenvalue +1 or -1, i.e., $M(S, (e_i))$ is diagonal with diagonal values of +1 or -1.
- $\mathbb{F} = \mathbb{R}$ and S an isometry $\iff \exists$ an orthonormal basis (e_i) s.t. $M(S, (e_i))$ is block diagonal with 1×1 blocks of value +1 or -1, and with 2×2 blocks of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta \in (0, \pi)$.

Arbitrary Operators on V:
Every $T \in L(V)$ can be *Polar Decomposed* as $T = S\sqrt{T^*T}$; S is an isometry and $\sqrt{T^*T}$ is a positive operator. Furthermore, the *Singular Value Decomposition* tells that $\exists$ orthonormal bases (e_i) and (f_i) of V s.t. $\forall v \in V$,

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n, \quad \text{that is,} \quad M(T, (e_i), (f_i)) = \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{bmatrix}, \tag{1}$$

where s_i are the *singular values* of T, given by the eigenvalues of the positive operator $\sqrt{T^*T}$. Each s_i is real and non-negative.

Eigenvector Orthonormality:
The complex spectral theorem tells that normality is equivalent to being *unitarily diagonalizable*. Of course, many non-normal matrices are diagonalizable, only they aren't unitarily diagonalizable, i.e., their eigenvectors aren't orthonormal. In fact, the case of an orthonormal basis of eigenvectors is obviously a rare thing- in the space of matrices, the normal matrices have measure 0.

The matrices that can't be diagonalized also have measure 0; these are the matrices with non-trivial Jordan blocks, i.e., with non-trivial generalized eigenvectors. Usually, the matrix of $T \in L(V)$ has $\dim(V)$ distinct eigenvalues in $\mathbb{C}$. With non-trivial Jordan blocks, some of these points in the plane have to exactly coencide, which is obviously a rare thing.