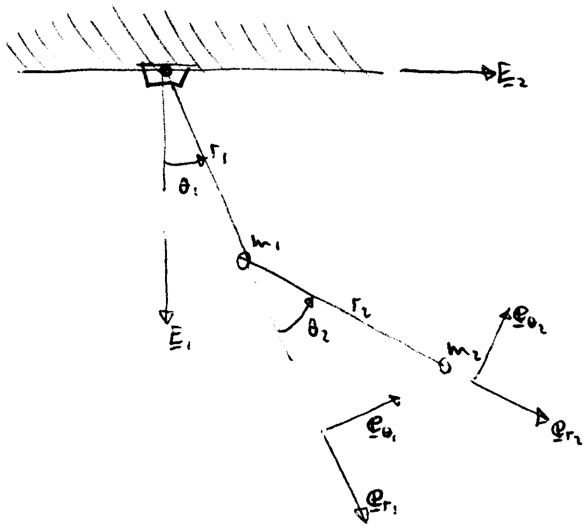

Lagrange's Equations for a 2-chain.

Patch Kessler

Unit Vectors

$$\mathbf{e}_{r_1} = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix}_{\mathbf{E}_i} \qquad \mathbf{e}_{\theta_1} = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}_{\mathbf{E}_i} \qquad \mathbf{e}_{r_2} = \begin{bmatrix} \cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) \end{bmatrix}_{\mathbf{E}_i} \qquad \mathbf{e}_{\theta_2} = \begin{bmatrix} -\sin(\theta_1+\theta_2) \\ \cos(\theta_1+\theta_2) \end{bmatrix}_{\mathbf{E}_i}$$

$$\mathbf{e}_{r_1}\cdot\mathbf{e}_{r_2} = \cos\theta_2 \qquad \mathbf{e}_{r_1}\cdot\mathbf{e}_{\theta_2} = \cos\left(\tfrac{\pi}{2}+\theta_2\right) = -\sin\theta_2$$

$$\mathbf{e}_{\theta_1}\cdot\mathbf{e}_{\theta_2} = \cos\theta_2 \qquad \mathbf{e}_{r_2}\cdot\mathbf{e}_{\theta_1} = \cos\left(\tfrac{\pi}{2}-\theta_2\right) = \sin\theta_2$$

Position and Velocity

$$\mathbf{x}_1 = r_1\mathbf{e}_{r_1} \qquad \mathbf{x}_2 = \mathbf{x}_1 + r_2\mathbf{e}_{r_2}$$

$$\dot{\mathbf{x}}_1 = \dot{r}_1\mathbf{e}_{r_1} + r_1\dot{\theta}_1\mathbf{e}_{\theta_1} \qquad \dot{\mathbf{x}}_2 = \dot{\mathbf{x}}_1 + \dot{r}_2\mathbf{e}_{r_2} + r_2(\dot{\theta}_1+\dot{\theta}_2)\mathbf{e}_{\theta_2}$$

Kinetic Energy

$$T = \frac{1}{2}m_1\dot{\mathbf{x}}_1\cdot\dot{\mathbf{x}}_1 + \frac{1}{2}m_2\dot{\mathbf{x}}_2\cdot\dot{\mathbf{x}}_2$$

$$= \frac{m_1}{2}\left(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2\right) + \frac{m_2}{2}\left(\left(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2\right) + \left(\dot{r}_2^2 + r_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2\right) + 2\dot{r}_2\overbrace{\left(\dot{r}_1\underbrace{\mathbf{e}_{r_1}\cdot\mathbf{e}_{r_2}}_{\cos\theta_2} + r_1\dot{\theta}_1\underbrace{\mathbf{e}_{r_2}\cdot\mathbf{e}_{\theta_1}}_{\sin\theta_2}\right)}^{\dot{\mathbf{x}}_1\cdot\mathbf{e}_{r_2}}\right)$$

$$+ \frac{m_2}{2}\cdot 2r_2\left(\dot{\theta}_1+\dot{\theta}_2\right)\overbrace{\left(\dot{r}_1\underbrace{\mathbf{e}_{r_1}\cdot\mathbf{e}_{\theta_2}}_{-\sin\theta_2} + r_1\dot{\theta}_1\underbrace{\mathbf{e}_{\theta_1}\cdot\mathbf{e}_{\theta_2}}_{\cos\theta_2}\right)}^{\dot{\mathbf{x}}_1\cdot\mathbf{e}_{\theta_2}}$$

$$= \frac{m_1+m_2}{2}\left(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2\right) + \frac{m_2}{2}\left(\dot{r}_2^2 + r_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2\right) + \cos\theta_2\left(m_2\dot{r}_1\dot{r}_2 + m_2 r_1 r_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)\right)$$

$$+ \sin\theta_2\left(m_2 r_1\dot{r}_2\dot{\theta}_1 - m_2\dot{r}_1 r_2\left(\dot{\theta}_1+\dot{\theta}_2\right)\right)$$

Derivatives

$$\frac{\partial T}{\partial \dot r_1} = (m_1 + m_2)\dot r_1 + m_2 \dot r_2 \cos\theta_2 - m_2 r_2 (\dot\theta_1 + \dot\theta_2)\sin\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot r_1}\right) = (m_1 + m_2)\ddot r_1 + m_2 \ddot r_2 \cos\theta_2 - m_2 \dot r_2 \dot\theta_2 \sin\theta_2 - m_2 \dot r_2 (\dot\theta_1 + \dot\theta_2)\sin\theta_2 - m_2 r_2 (\ddot\theta_1 + \ddot\theta_2)\sin\theta_2 - m_2 r_2 \dot\theta_2 (\dot\theta_1 + \dot\theta_2)\cos\theta_2$$

$$\frac{\partial T}{\partial r_1} = (m_1 + m_2) r_1 \dot\theta_1^2 + m_2 r_2 \dot\theta_1 (\dot\theta_1 + \dot\theta_2)\cos\theta_2 + m_2 \dot r_2 \dot\theta_1 \sin\theta_2$$

Also, $\dfrac{\partial \underline{x}_1}{\partial r_1} = \underline{e}_{r_1}$, $\dfrac{\partial \underline{x}_2}{\partial r_1} = \underline{e}_{r_1}$

$$\frac{\partial T}{\partial \dot r_2} = m_2 \dot r_2 + m_2 \dot r_1 \cos\theta_2 + m_2 r_1 \dot\theta_1 \sin\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot r_2}\right) = m_2 \ddot r_2 + m_2 \ddot r_1 \cos\theta_2 - m_2 \dot r_1 \dot\theta_2 \sin\theta_2 + m_2 \dot r_1 \dot\theta_1 \sin\theta_2 + m_2 r_1 \ddot\theta_1 \sin\theta_2 + m_2 r_1 \dot\theta_1 \dot\theta_2 \cos\theta_2$$

$$\frac{\partial T}{\partial r_2} = m_2 r_2 (\dot\theta_1 + \dot\theta_2)^2 + m_2 r_1 \dot\theta_1 (\dot\theta_1 + \dot\theta_2)\cos\theta_2 - m_2 \dot r_1 (\dot\theta_1 + \dot\theta_2)\sin\theta_2$$

Also, $\dfrac{\partial \underline{x}_1}{\partial r_2} = \underline{0}$, $\dfrac{\partial \underline{x}_2}{\partial r_2} = \underline{e}_{r_2}$

$$\frac{\partial T}{\partial \dot\theta_1} = (m_1 + m_2) r_1^2 \dot\theta_1 + m_2 r_2^2 (\dot\theta_1 + \dot\theta_2) + m_2 r_1 r_2 (2\dot\theta_1 + \dot\theta_2)\cos\theta_2 + m_2 r_1 \dot r_2 \sin\theta_2 - m_2 \dot r_1 r_2 \sin\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_1}\right) = (m_1 + m_2)\cdot 2 r_1 \dot r_1 \dot\theta_1 + (m_1 + m_2) r_1^2 \ddot\theta_1 + m_2 \cdot 2 r_2 \dot r_2 (\dot\theta_1 + \dot\theta_2) + m_2 r_2^2 (\ddot\theta_1 + \ddot\theta_2)$$
$$+ m_2 (\dot r_1 r_2 + r_1 \dot r_2)(2\dot\theta_1 + \dot\theta_2)\cos\theta_2 + m_2 r_1 r_2 (2\ddot\theta_1 + \ddot\theta_2)\cos\theta_2 - m_2 r_1 r_2 \dot\theta_2 (2\dot\theta_1 + \dot\theta_2)\sin\theta_2$$
$$+ m_2 (r_1 \ddot r_2 - \ddot r_1 r_2)\sin\theta_2 + m_2 (r_1 \dot r_2 - \dot r_1 r_2)\dot\theta_2 \cos\theta_2$$

$$\frac{\partial T}{\partial \theta_1} = 0$$

Also, $\dfrac{\partial \underline{x}_1}{\partial \theta_1} = r_1 \underline{e}_{\theta_1}$, $\dfrac{\partial \underline{x}_2}{\partial \theta_1} = r_1 \underline{e}_{\theta_1} + r_2 \underline{e}_{\theta_2}$

$$\frac{\partial T}{\partial \dot\theta_2} = m_2 r_2^2 (\dot\theta_1 + \dot\theta_2) + m_2 r_1 r_2 \dot\theta_1 \cos\theta_2 - m_2 \dot r_1 r_2 \sin\theta_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_2}\right) = m_2 \cdot 2 r_2 \dot r_2 (\dot\theta_1 + \dot\theta_2) + m_2 r_2^2 (\ddot\theta_1 + \ddot\theta_2)$$
$$+ m_2 (\dot r_1 r_2 + r_1 \dot r_2)\dot\theta_1 \cos\theta_2 + m_2 r_1 r_2 \ddot\theta_1 \cos\theta_2 - m_2 r_1 r_2 \dot\theta_1 \dot\theta_2 \sin\theta_2$$
$$- m_2 \ddot r_1 r_2 \sin\theta_2 - m_2 \dot r_1 \dot r_2 \sin\theta_2 - m_2 \dot r_1 r_2 \dot\theta_2 \cos\theta_2$$

$$\frac{\partial T}{\partial \theta_2} = \cos\theta_2 \left(m_2 r_1 \dot r_2 \dot\theta_1 - m_2 \dot r_1 r_2 (\dot\theta_1 + \dot\theta_2)\right) - \sin\theta_2 \left(m_2 \dot r_1 \dot r_2 + m_2 r_1 r_2 \dot\theta_1 (\dot\theta_1 + \dot\theta_2)\right)$$

And $\frac{\partial \underline{x}_1}{\partial \theta_2} = \underline{0}$, $\frac{\partial \underline{x}_2}{\partial \theta_2} = r_2 \underline{e}_{\theta_2}$

Forces

let $\underline{f}_1 = m_1 g \underline{E}_1 + c_1 \underline{e}_{r_1} - c_2 \underline{e}_{r_2}$

then

$$\underline{f}_1 \cdot \left(\frac{\partial \underline{x}_1}{\partial r_1} = \underline{e}_{r_1}\right) = m_1 g \cos\theta_1 + c_1 - c_2 \cos\theta_2$$

$$\underline{f}_1 \cdot \left(\frac{\partial \underline{x}_1}{\partial r_2} = \underline{0}\right) = 0$$

$$\underline{f}_1 \cdot \left(\frac{\partial \underline{x}_1}{\partial \theta_1} = r_1 \underline{e}_{\theta_1}\right) = -m_1 g r_1 \sin\theta_1 - r_1 c_2 \sin\theta_2$$

$$\underline{f}_1 \cdot \left(\frac{\partial \underline{x}_1}{\partial \theta_2} = \underline{0}\right) = 0$$

let $\underline{f}_2 = m_2 g \underline{E}_1 + c_2 \underline{e}_{r_2}$

then,

$$\underline{f}_2 \cdot \left(\frac{\partial \underline{x}_2}{\partial r_1} = \underline{e}_{r_1}\right) = m_2 g \cos\theta_1 + c_2 \cos\theta_2$$

$$\underline{f}_2 \cdot \left(\frac{\partial \underline{x}_2}{\partial r_2} = \underline{e}_{r_2}\right) = m_2 g \cos(\theta_1 + \theta_2) + c_2$$

$$\underline{f}_2 \cdot \left(\frac{\partial \underline{x}_2}{\partial \theta_1} = r_1 \underline{e}_{\theta_1} + r_2 \underline{e}_{\theta_2}\right) = c_2 r_1 \sin\theta_2 - m_2 g \left(r_1 \sin\theta_1 + r_2 \sin(\theta_1 + \theta_2)\right)$$

$$\underline{f}_2 \cdot \left(\frac{\partial \underline{x}_2}{\partial \theta_2} = r_2 \underline{e}_{\theta_2}\right) = -m_2 g r_2 \sin(\theta_1 + \theta_2)$$

The case of constraints: let r_1, r_2 be constant.

$r_1)$

$$\begin{aligned}&-m_2 r_2(\ddot\theta_1+\ddot\theta_2)\sin\theta_2 \;\cancel{-m_2 r_2\dot\theta_2(\dot\theta_1+\dot\theta_2)\cos\theta_2} \;\to\; -m_2 r_2(\dot\theta_1+\dot\theta_2)^2\cos\theta_2\\ &-(m_1+m_2)r_1\dot\theta_1^2 \;\cancel{-m_2 r_2\dot\theta_1(\dot\theta_1+\dot\theta_2)\cos\theta_2} \;=\; c_1 + (m_1+m_2)g\cos\theta_1\end{aligned}$$

$r_2)$

$$\begin{aligned}&m_2 r_1\ddot\theta_1\sin\theta_2 \;\cancel{+m_2 r_1\dot\theta_1\dot\theta_2\cos\theta_2} \;=\; c_2 + m_2 g\cos(\theta_1+\theta_2)\\ &-m_2 r_2(\dot\theta_1+\dot\theta_2)^2 \;\cancel{-m_2 r_1\dot\theta_1(\dot\theta_1+\dot\theta_2)\cos\theta_2}\\ &\to\; -m_2 r_1\dot\theta_1^2\cos\theta_2\end{aligned}$$

$\theta_1)$

$$\begin{aligned}&(m_1+m_2)r_1^2\ddot\theta_1 + m_2 r_2^2(\ddot\theta_1+\ddot\theta_2)\\ &+m_2 r_1 r_2(2\ddot\theta_1+\ddot\theta_2)\cos\theta_2 - m_2 r_1 r_2\dot\theta_2(2\dot\theta_1+\dot\theta_2)\sin\theta_2 = -(m_1+m_2)g r_1\sin\theta_1 - m_2 g r_2\sin(\theta_1+\theta_2)\end{aligned}$$

$\theta_2)$

$$\begin{aligned}&m_2 r_2^2(\ddot\theta_1+\ddot\theta_2) + m_2 r_1 r_2\ddot\theta_1\cos\theta_2 - m_2 r_1 r_2\dot\theta_1\dot\theta_2\sin\theta_2 = -m_2 g r_2\sin(\theta_1+\theta_2)\\ &+m_2 r_1 r_2\dot\theta_1(\dot\theta_1+\dot\theta_2)\sin\theta_2\end{aligned}$$

$\theta_1) + \theta_2) \;\longrightarrow$

$$\begin{bmatrix}(m_1+m_2)r_1^2 + m_2 r_2^2 + 2m_2 r_1 r_2\cos\theta_2 & m_2 r_2^2 + m_2 r_1 r_2\cos\theta_2\\ m_2 r_2^2 + m_2 r_1 r_2\cos\theta_2 & m_2 r_2^2\end{bmatrix}\begin{bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{bmatrix} = \begin{bmatrix}-(m_1+m_2)g r_1\sin\theta_1 - m_2 g r_2\sin(\theta_1+\theta_2) + m_2 r_1 r_2\dot\theta_2(2\dot\theta_1+\dot\theta_2)\sin\theta_2\\ -m_2 g r_2\sin(\theta_1+\theta_2) + m_2 r_1 r_2\dot\theta_1\dot\theta_2\sin\theta_2 - m_2 r_1 r_2\dot\theta_1(\dot\theta_1+\dot\theta_2)\sin\theta_2\end{bmatrix}$$

If we wish to know the constraint forces c_1 and c_2 we can now solve for them using $r_1)$ and $r_2)$.

Approach 2

Same system as before, with the same unit vectors.
This time, keep r_1, r_2 constant from the very beginning.

Position & Velocity

$$\underline{x}_1 = r_1 \underline{e}_{r_1} \qquad \underline{x}_2 = \underline{x}_1 + r_2 \underline{e}_{r_2}$$

$$\dot{\underline{x}}_1 = r_1 \dot{\theta}_1 \underline{e}_{\theta_1} \qquad \dot{\underline{x}}_2 = r_1 \dot{\theta}_1 \underline{e}_{\theta_1} + r_2 (\dot{\theta}_1 + \dot{\theta}_2) \underline{e}_{\theta_2}$$

Kinetic Energy

$$T = \frac{1}{2} m_1 \dot{\underline{x}}_1 \cdot \dot{\underline{x}}_1 + \frac{1}{2} m_2 \dot{\underline{x}}_2 \cdot \dot{\underline{x}}_2$$

$$= \frac{m_1}{2} r_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} \left(r_1^2 \dot{\theta}_1^2 + 2 r_1 r_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \cos\theta_2 + r_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \right)$$

Derivatives

$$\frac{\partial T}{\partial \dot{\theta}_1} = m_1 r_1^2 \dot{\theta}_1 + m_2 r_1^2 \dot{\theta}_1 + m_2 r_1 r_2 (2\dot{\theta}_1 + \dot{\theta}_2) \cos\theta_2 + m_2 r_2^2 (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) r_1^2 \ddot{\theta}_1 + m_2 r_1 r_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) \cos\theta_2 - m_2 r_1 r_2 \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2) \sin\theta_2 + m_2 r_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2)$$

$$\frac{\partial T}{\partial \theta_1} = 0 \qquad \text{And} \quad \frac{\partial \underline{x}_1}{\partial \theta_1} = r_1 \underline{e}_{\theta_1} \qquad \frac{\partial \underline{x}_2}{\partial \theta_1} = r_1 \underline{e}_{\theta_1} + r_2 \underline{e}_{\theta}$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = m_2 r_1 r_2 \dot{\theta}_1 \cos\theta_2 + m_2 r_2^2 (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = m_2 r_1 r_2 \ddot{\theta}_1 \cos\theta_2 - m_2 r_1 r_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2 + m_2 r_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2)$$

$$\frac{\partial T}{\partial \theta_2} = -m_2 r_1 r_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \sin\theta_2 \qquad \text{And} \quad \frac{\partial \underline{x}_1}{\partial \theta_2} = \underline{0}, \quad \frac{\partial \underline{x}_2}{\partial \theta_2} = r_2 \underline{e}_{\theta_2}$$

Agrees with final result from A4 if $\underline{F}_1 = m_1 g \underline{E}_1$ and $\underline{F}_2 = m_2 g \underline{E}_1$

$$\longrightarrow \begin{bmatrix} (m_1 + m_2) r_1^2 + 2 m_2 r_1 r_2 \cos\theta_2 + m_2 r_2^2 & m_2 r_1 r_2 \cos\theta_2 + m_2 r_2^2 \\ m_2 r_1 r_2 \cos\theta_2 + m_2 r_2^2 & m_2 r_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} m_2 r_1 r_2 \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2) \sin\theta_2 \\ + m_2 r_1 r_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2 - m_2 r_1 r_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \sin\theta_2 \end{bmatrix} + \begin{bmatrix} \underline{F}_1 \cdot \frac{\partial \underline{x}_1}{\partial \theta_1} + \underline{F}_2 \cdot \frac{\partial \underline{x}_2}{\partial \theta_1} \\ \underline{F}_1 \cdot \frac{\partial \underline{x}}{\partial \theta_2} + \underline{F}_2 \cdot \frac{\partial \underline{x}_2}{\partial \theta_2} \end{bmatrix}$$