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The Serret-Frenet Triad

Let $\mathbf{x}(s)$ be a curve in E^3, and suppose that s is an arc-length parameter (i.e., that $\|\frac{d\mathbf{x}}{ds}\| = 1$). At any point on the curve, we define the *tangent vector* $\mathbf{t} = \frac{d\mathbf{x}}{ds}$, the *curvature* $\kappa = \|\frac{d\mathbf{t}}{ds}\|$, the *normal vector* $\mathbf{n} = \frac{1}{\kappa}\frac{d\mathbf{t}}{ds}$, and finally the *binormal vector* $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. Although plenty of vectors besides $\mathbf{t}$ and $\mathbf{n}$ are geometrically tangent and normal to the curve, our definitions allow us to say "the tangent vector" and "the normal vector" without ambiguity. Notice that $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ are orthonormal. These vectors are referred to as the *Serret-Frenet* triad.

1 The Serret-Frenet Triad Derivatives

We know that $\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}$, but how about $\frac{d\mathbf{n}}{ds}$ and $\frac{d\mathbf{t}}{ds}$? Because $\mathbf{n}\cdot\mathbf{n} = 1 \implies \frac{d\mathbf{n}}{ds}\cdot\mathbf{n} = 0$, it must be that $\frac{d\mathbf{n}}{ds} = \alpha\mathbf{t} + \beta\mathbf{b}$, where α and β are scalars. Computing $\frac{d\mathbf{b}}{ds}$, we find that $\frac{d\mathbf{b}}{ds} = \frac{d\mathbf{t}}{ds}\times\mathbf{n} + \mathbf{t}\times\frac{d\mathbf{n}}{ds} = -\beta\mathbf{n}$. The common name for β is the *torsion* of the curve, and it is usually denoted by τ. Finally, note that $\alpha = \frac{d\mathbf{n}}{ds}\cdot\mathbf{t} = \frac{d}{ds}(\mathbf{n}\cdot\mathbf{t}) - \mathbf{n}\cdot\frac{d\mathbf{t}}{ds} = -\kappa$. These formulae can be summarized in a single matrix equation:

$$\frac{d}{ds}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \tag{1}$$

Now consider $\mathbf{x} = x_t\mathbf{t} + x_n\mathbf{n} + x_b\mathbf{b}$, which can be expressed as the following formal product

$$\mathbf{x} = \begin{bmatrix}\mathbf{t} & \mathbf{n} & \mathbf{b}\end{bmatrix}\begin{bmatrix}x_t\\ x_n\\ x_b\end{bmatrix}. \tag{2}$$

The column matrix on the right is called *the matrix of* $\mathbf{x}$ *with respect to the Serret-Frenet triad*. We obtain the matrix of $\frac{d\mathbf{x}}{ds}$ with respect to the Serret-Frenet triad by differentiating (2), (using the transpose of (1))

$$\frac{d\mathbf{x}}{ds} = \begin{bmatrix}\mathbf{t} & \mathbf{n} & \mathbf{b}\end{bmatrix}\left(\begin{bmatrix}0 & -\kappa & 0\\ \kappa & 0 & -\tau\\ 0 & \tau & 0\end{bmatrix}\begin{bmatrix}x_t\\ x_n\\ x_b\end{bmatrix} + \frac{d}{ds}\begin{bmatrix}x_t\\ x_n\\ x_b\end{bmatrix}\right). \tag{3}$$

The skew-symmetric matrix in (3) corresponds to a cross product as follows

$$\begin{bmatrix}\mathbf{t} & \mathbf{n} & \mathbf{b}\end{bmatrix}\begin{bmatrix}0 & -\kappa & 0\\ \kappa & 0 & -\tau\\ 0 & \tau & 0\end{bmatrix}\begin{bmatrix}x_t\\ x_n\\ x_b\end{bmatrix} = (\begin{bmatrix}\mathbf{t} & \mathbf{n} & \mathbf{b}\end{bmatrix}\begin{bmatrix}\tau\\ 0\\ \kappa\end{bmatrix}) \times (\begin{bmatrix}\mathbf{t} & \mathbf{n} & \mathbf{b}\end{bmatrix}\begin{bmatrix}x_t\\ x_n\\ x_b\end{bmatrix} = \mathbf{x}). \tag{4}$$

It follows that if x_t, x_n, and x_b are s independent (as they are when $\mathbf{x}$ equals $\mathbf{t}$, $\mathbf{n}$, or $\mathbf{b}$), then as s increases, $\mathbf{x}$ spins about the vector $\tau\mathbf{t} + \kappa\mathbf{b}$, at a rate proportional to $\sqrt{\tau^2 + \kappa^2}$. This is totally awesome.

1.1 Non Arc-Length Parameters

We use s to denote arc-length parameters, and t to denote non arc-length parameters, (i.e., those for which $\|\frac{d\mathbf{x}}{dt}\| \neq 1$). Letting $u^{[k]}$ denote $\frac{d^k u}{dt^k}$, and letting p denote $\|\mathbf{x}^{[1]}\|$, we note that

$$p\mathbf{t} = \mathbf{x}^{[1]}, \qquad p^2\kappa\mathbf{n} = \mathbf{x}^{[2]} - \mathbf{t}(\mathbf{t}\cdot\mathbf{x}^{[2]}), \qquad \mathbf{b} = \mathbf{t}\times\mathbf{n}, \qquad p^2\kappa\tau = \mathbf{b}\cdot\mathbf{x}^{[3]} \tag{5}$$

From these we can compute $\mathbf{t}$, p, $\mathbf{n}$, κ, $\mathbf{b}$, and finally τ. Finding t_2 so that the length of $\mathbf{x}(t)$ from some t_1 to t_2 has a desired value L can be done by using Newton's method on the function $L(t_2)$. Note that

$$L(t_2) = \int_{t_1}^{t_2} p\, dt \text{ and that } \frac{dL}{dt_2} = p. \tag{6}$$