The Serret-Frenet Triad

Let \( x(s) \) be a curve in \( E^3 \), and suppose that \( s \) is an arc-length parameter (i.e., that \( \| dx \| = 1 \)). At any point on the curve, we define the tangent vector \( t = \frac{dx}{ds} \), the curvature \( \kappa = \| \frac{dt}{ds} \| \), the normal vector \( n = \frac{1}{\kappa} \frac{dt}{ds} \), and finally the binormal vector \( b = t \times n \). Although plenty of vectors besides \( t \) and \( n \) are geometrically tangent and normal to the curve, our definitions allow us to say “the tangent vector” and “the normal vector” without ambiguity. Notice that \( t, n, \) and \( b \) are orthonormal. These vectors are referred to as the Serret-Frenet triad.

1 The Serret-Frenet Triad Derivatives

We know that \( \frac{dt}{ds} = \kappa n \), but how about \( \frac{dn}{ds} \) and \( \frac{db}{ds} \)? Because \( n \cdot n = 1 \implies \frac{dn}{ds} \cdot n = 0 \), it must be that \( \frac{dn}{ds} = \alpha t + \beta b \), where \( \alpha \) and \( \beta \) are scalars. Computing \( \frac{db}{ds} \), we find that \( \frac{db}{ds} = \frac{dt}{ds} \times n + t \times \frac{dn}{ds} = -\beta n \). The common name for \( \beta \) is the torsion of the curve, and it is usually denoted by \( \tau \). Finally, note that

\[
\alpha = \frac{dn}{ds} \cdot t = \frac{ds}{ds} (n \cdot t) - n \cdot \frac{dt}{ds} = -\kappa.
\]

These formulae can be summarized in a single matrix equation:

\[
\frac{d}{ds} \begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}
\]

(1)

Now consider \( x = x_t t + x_n n + x_b b \), which can be expressed as the following formal product

\[
x = [t \ n \ b] \begin{bmatrix} x_t \\ x_n \\ x_b \end{bmatrix}.
\]

(2)

The column matrix on the right is called the matrix of \( x \) with respect to the Serret-Frenet triad. We obtain the matrix of \( \frac{dx}{ds} \) with respect to the Serret-Frenet triad by differentiating (2), (using the transpose of (1))

\[
\frac{dx}{ds} = [t \ n \ b] \left( \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} x_t \\ x_n \\ x_b \end{bmatrix} + \frac{d}{ds} \begin{bmatrix} x_t \\ x_n \\ x_b \end{bmatrix} \right).
\]

(3)

The skew-symmetric matrix in (3) corresponds to a cross product as follows

\[
[t \ n \ b] \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} x_t \\ x_n \\ x_b \end{bmatrix} = ([t \ n \ b] \begin{bmatrix} \tau \\ 0 \\ \kappa \end{bmatrix}) \times ([t \ n \ b] \begin{bmatrix} x_t \\ x_n \\ x_b \end{bmatrix}) = x.
\]

(4)

It follows that if \( x_t, x_n, \) and \( x_b \) are s independent (as they are when \( x \) equals \( t, n, \) or \( b \)), then as \( s \) increases, \( x \) spins about the vector \( \tau t + \kappa b \), at a rate proportional to \( \sqrt{\tau^2 + \kappa^2} \). This is totally awesome.

1.1 Non Arc-Length Parameters

We use \( s \) to denote arc-length parameters, and \( t \) to denote non arc-length parameters, (i.e., those for which \( \| \frac{dx}{dt} \| \neq 1 \)). Letting \( u^{[k]} \) denote \( \frac{dx}{dt^k} \), and letting \( p \) denote \( \| x^{[1]} \| \), we note that

\[
pt = x^{[1]}, \quad p^2 \kappa n = x^{[2]} - t(t \cdot x^{[2]}), \quad b = t \times n, \quad p^2 \kappa \tau = b \cdot x^{[3]}
\]

(5)

From these we can compute \( t, p, n, \kappa, b, \) and finally \( \tau \). Finding \( t_2 \) so that the length of \( x(t) \) from some \( t_1 \) to \( t_2 \) has a desired value \( L \) can be done by using Newton's method on the function \( L(t_2) \). Note that

\[
L(t_2) = \int_{t_1}^{t_2} p \, dt \quad \text{and that} \quad \frac{dL}{dt_2} = p.
\]

(6)