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Notes on Integrability

Disclaimer: My conception of what it means for something to be *integrable* is still evolving, and so these notes may contain errors. If you find any errors or if you'd like to discuss this topic with me, please send me email at `watchwrk@me.berkeley.edu`

Let V be an inner product space over $\mathbb{R}$ and let the word *vector field* refer to a mapping from V to V.

Definition 1: The vector field $\mathbf{v}$ is called *integrable* on the open connected $B \subset V$ if the line integral

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{x} \tag{1}$$

is independent of the path in B along which it is evaluated.

Definition 2: If $\Psi : V \longrightarrow \mathbb{R}$, then the *gradient* of Ψ is defined as the mapping $\nabla\Psi : V \longrightarrow V$ which satisfies

$$\nabla\Psi(\mathbf{x}) \cdot \mathbf{c} = \frac{d}{ds}\Psi(\mathbf{x} + s\mathbf{c})\Big|_{s=0} \tag{2}$$

for all $\mathbf{c} \in V$.

If $\Psi : V \longrightarrow \mathbb{R}$ and $\{\mathbf{e}_i\}$ is an orthonormal basis of V, then

$$\nabla\Psi = (\nabla\Psi \cdot \mathbf{e}_i)\mathbf{e}_i = \left(\frac{d}{ds}\Psi(\mathbf{x} + s\mathbf{e}_i)\Big|_{s=0}\right)\mathbf{e}_i = \frac{\partial\Psi}{\partial x_i}\mathbf{e}_i, \tag{3}$$

where $\frac{\partial\Psi}{\partial x_i}$ in the last term denotes $\frac{d}{ds}\Psi(\mathbf{x} + s\mathbf{e}_i)\big|_{s=0}$. Thus we obtain the usual formula for the gradient.

Theorem: The vector field $\mathbf{v}$ is integrable on B if and only if there exists a $\Psi : B \longrightarrow \mathbb{R}$ for which $\nabla\Psi = \mathbf{v}$.

Proof: Let $\mathbf{x}(s)$ be any arc-length parameterized path in B, from $\mathbf{a} = \mathbf{x}(s_a)$ to $\mathbf{b} = \mathbf{x}(s_b)$. If a $\Psi : B \longrightarrow \mathbb{R}$ exists for which $\nabla\Psi = \mathbf{v}$, then the line integral of $\mathbf{v}$ along the path $\mathbf{x}(s)$ is given by

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{x} = \int_{\mathbf{a}}^{\mathbf{b}} \nabla\Psi \cdot d\mathbf{x} = \int_{s_a}^{s_b} \nabla\Psi(\mathbf{x}(s)) \cdot \mathbf{t}(s)ds = \int_{s_a}^{s_b} \frac{d}{d\tilde{s}}\Psi(\mathbf{x}(s) + \tilde{s}\mathbf{t}(s))\Big|_{\tilde{s}=0} ds, \tag{4}$$

where $\mathbf{t}(s)$ after the second equality is the unit tangent vector to $\mathbf{x}(s)$, and where the third equality follows from (2). Noting that $\mathbf{x}(s) + \Delta s\ \mathbf{t}(s) \longrightarrow \mathbf{x}(s + \Delta s)$ as $\Delta s \longrightarrow 0$, we obtain

$$\begin{aligned}
\frac{d}{d\tilde{s}}\Psi(\mathbf{x}(s) + \tilde{s}\mathbf{t}(s))\Big|_{\tilde{s}=0} &= \lim_{\Delta s \to 0} \frac{\Psi(\mathbf{x}(s) + \Delta s\ \mathbf{t}(s)) - \Psi(\mathbf{x}(s))}{\Delta s} \\
&= \lim_{\Delta s \to 0} \frac{\Psi(\mathbf{x}(s + \Delta s)) - \Psi(\mathbf{x}(s))}{\Delta s} \\
&= \frac{d\Psi(\mathbf{x}(s))}{ds},
\end{aligned} \tag{5}$$

which allows us to write (4) as

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{x} = \int_{s_a}^{s_b} \frac{d\Psi(\mathbf{x}(s))}{ds} ds = \Psi(\mathbf{b}) - \Psi(\mathbf{a}). \tag{6}$$

This result is independent of the path $\mathbf{x}(s)$, as we wished to show.

Now suppose that the vector field $\mathbf{v}$ is integrable on $B \subset V$. The path independence of the line integral (1) allows us to define a $\Psi : B \longrightarrow \mathbb{R}$ according to

$$\Psi(\mathbf{x}) = \int_{\mathbf{a}}^{\mathbf{x}} \mathbf{v} \cdot d\mathbf{x}, \tag{7}$$

where $\mathbf{a}$ is some point in B. We need to show that $\nabla\Psi = \mathbf{v}$. Let $\{\mathbf{e}_i\}$ be an orthonormal basis for V, and note that

$$\Psi(\mathbf{x} + \varepsilon\mathbf{e}_i) - \Psi(\mathbf{x}) = \int_{\mathbf{x}}^{\mathbf{x}+\varepsilon\mathbf{e}_i} \mathbf{v} \cdot d\mathbf{x}. \tag{8}$$

The line integral is path independent and so for convenience we evaluate it along the straight line from $\mathbf{x}$ to $\mathbf{x} + \varepsilon\mathbf{e}_i$, (which is in the open set B for ε sufficiently small). We obtain

$$\Psi(\mathbf{x} + \varepsilon\mathbf{e}_i) - \Psi(\mathbf{x}) = \int_0^{\varepsilon} (\mathbf{v}(\mathbf{x}) \cdot \mathbf{e}_i + O(s))ds = \varepsilon\mathbf{v}(\mathbf{x}) \cdot \mathbf{e}_i + O(\varepsilon^2). \tag{9}$$

Dividing through by ε and taking the limit as $\varepsilon \longrightarrow 0$, we obtain

$$\left.\frac{d}{ds}\Psi(\mathbf{x} + s\mathbf{e}_i)\right|_{s=0} = \mathbf{v}(\mathbf{x}) \cdot \mathbf{e}_i. \tag{10}$$

which from (2) implies that

$$\nabla\Psi(\mathbf{x}) \cdot \mathbf{e}_i = \mathbf{v}(\mathbf{x}) \cdot \mathbf{e}_i. \tag{11}$$

This holds for every $\mathbf{e}_i$ and so it follows that $\nabla\Psi = \mathbf{v}$ as desired.$\square$

Solving Problems: The easiest way to show that some vector field $\mathbf{v}$ is non-integrable is to show that the integral (1) is path dependent. For instance, if V is the xy-plane and if $\mathbf{v} = [xy \;\; 0]^T$, then the integral (1) along the path consisting of a straight line from the origin to $[1 \;\; 0]^T$, and then another straight line from $[1 \;\; 0]^T$ to $[1 \;\; 1]^T$ is clearly zero. However the integral going in straight lines from the origin to $[0 \;\; 1]^T$ and then to $[1 \;\; 1]^T$ is nonzero.