Math128a

Spring 2006

Math128a Rehersal Problem

Solutions and Commentary

Part I

Our task is to generate a numerical solution to the IVP $\dot{y} = y$, $y(0) = 1$ using the following 2-step method

$$y(t-h) - y(t+h) = 2hy(t). \tag{1}$$

We write this method as the following difference equation

$$y_{n+2} - 2hy_{n+1} - y_n = 0. \tag{2}$$

and see that $y_n = \lambda^n$ is a solution when $\lambda = 0$ and when $\lambda = h \pm \sqrt{h^2+1}$, (plug λ^n into (2), factor out λ^n, and solve the remaining quadratic equation in λ). Writing $\lambda_1 = h + \sqrt{h^2+1}$ and $\lambda_2 = h - \sqrt{h^2+1}$, the *general* solution of (2) is given by

$$y_n = a\lambda_1^n + b\lambda_2^n \tag{3}$$

where a and b are arbitrary.

Part II

Let's find a and b in (3) so that $y_0 = 1$ and $y_1 = 1 + h + \frac{h^2}{2}$. (Note that $y_0 = e^0$ and that $y_1 = e^h + O(h^3)$, and so the *starting values* y_0 and y_1 agree to $O(h^3)$ with the actual solution e^t). Letting H denote $1+h+\frac{h^2}{2}$,

$$\left.\begin{array}{l} y_0 = 1 \longrightarrow a + b = 1 \\ y_1 = H \longrightarrow a\lambda_1 + b\lambda_2 = H \end{array}\right\} \longrightarrow \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ H \end{bmatrix} \longrightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ H \end{bmatrix} \frac{1}{\lambda_2 - \lambda_1}$$

Writing out a and b explicitly in terms of h we find that

$$a = \frac{1}{2} + \frac{1 + \frac{h^2}{2}}{2\sqrt{1+h^2}} \text{ and } b = \frac{1}{2} - \frac{1 + \frac{h^2}{2}}{2\sqrt{1+h^2}}. \tag{4}$$

Part III

Suppose we use (3) and (4) to construct a numerical solution y_n to our IVP over the *fixed* interval $[0, T]$. We'd like to know how this numerical solution differs from the actual solution at T. In particular, what is the limit of $y_N - e^T$ as $N \longrightarrow \infty$? (T is fixed, N gets big, and $h = \frac{T}{N}$ gets small.) Before going further, take a step back and realize that we are applying the 2-step method

$$y(t+h) - y(t-h) = 2hf(y) \tag{5}$$

to the IVP

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= 1. \end{aligned} \tag{6}$$

The method is *Consistent*, with $O(h^3)$ local truncation error. The method is *Stable* because when f is equal to zero, the resulting difference equation $y_{n+2} - y_n = 0$ has an associated characteristic equation $r^2 - 1 = 0$ with roots ± 1. Remember that

Consistency + *Stability* = *Convergence*

when f is sufficiently smooth, and so for our problem (where $f(y) = y$), the numerical solution will converge to the actual solution on the fixed interval $[0, T]$. In particular, $y_N - e^T \longrightarrow 0$ as $N \longrightarrow \infty$.