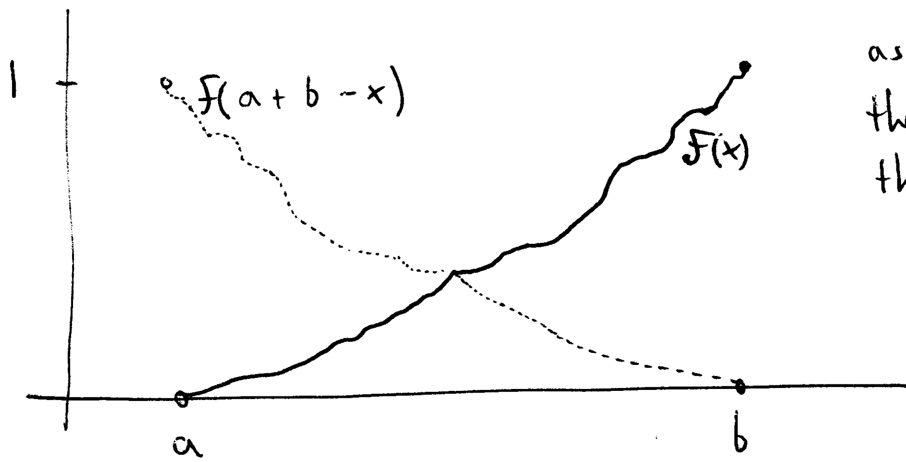
Concept From the Quiz

Let $F(x)$ be a continuous increasing function that increases from 0 to 1 on the interval $[a, b]$. Note that $F(x)$ need not be smooth. I have plotted a sample $F(x)$ below, as well as $F(a+b-x)$, which is the reflection of $F(x)$ about the vertical line $x = \frac{a+b}{2}$.



Of course $F(x) \leq 1$ and $F(a+b-x) \leq 1$ on $[a,b]$, so $F(x) \cdot F(a+b-x) \leq 1 \;\; \forall x \in [a,b]$, but we can do better than this.

<u>Claim</u>: $F(x) \cdot F(a+b-x) \leq \left(F\left(\frac{a+b}{2}\right)\right)^2$ for all $x \in [a,b]$.

<u>Proof</u>: Suppose the claim is false, so that for some $\tilde{x} \in [a, \frac{a+b}{2})$,

$$F(\tilde{x}) \cdot F(a+b-\tilde{x}) > F\left(\tfrac{a+b}{2}\right) \cdot F\left(\tfrac{a+b}{2}\right)$$

Because F is increasing, we know that $F(\tilde{x}) < F\left(\frac{a+b}{2}\right)$, and so

$$\cancel{F\left(\tfrac{a+b}{2}\right)} F(a+b-\tilde{x}) > F(\tilde{x}) \cdot F(a+b-\tilde{x}) > \cancel{F\left(\tfrac{a+b}{2}\right)} \cdot F\left(\tfrac{a+b}{2}\right)$$

$$\longrightarrow F(a+b-\tilde{x}) > F\left(\tfrac{a+b}{2}\right)$$

which contradicts the hypothesis that F is increasing. $\square$