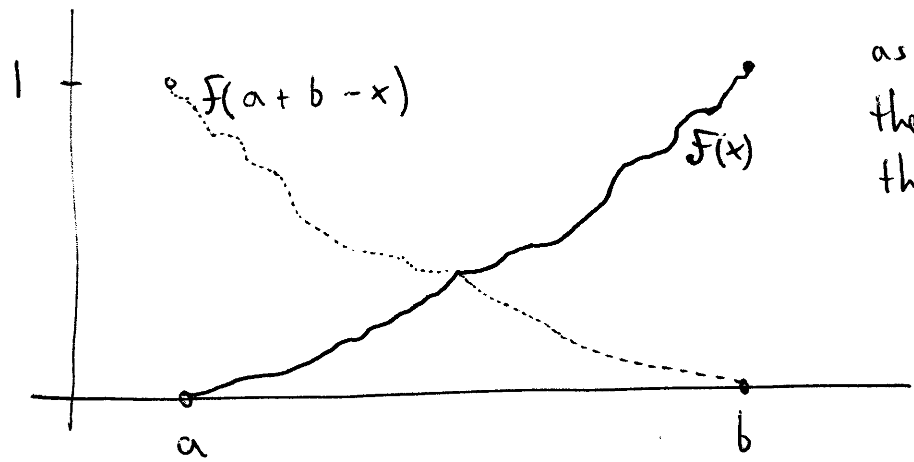
Concept From the Quiz

Let $f(x)$ be a continuous increasing function that increases from 0 to 1 on the interval $[a, b]$. Note that $f(x)$ need not be smooth. I have plotted a sample $f(x)$ below, as well as $f(a+b-x)$, which is the reflection of $f(x)$ about the vertical line $x = \frac{a+b}{2}$.



Of course $f(x) \leq 1$ and $f(a+b-x) \leq 1$ on $[a,b]$, so $f(x) \cdot f(a+b-x) \leq 1 \ \forall x \in [a,b]$, but we can do better than this.

<u>Claim</u>: $f(x) \cdot f(a+b-x) \leq \left(f\left(\frac{a+b}{2}\right)\right)^2$ for all $x \in [a,b]$.

<u>Proof</u>: Suppose the claim is false, so that for some $\tilde{x} \in [a, \frac{a+b}{2})$,

$$f(\tilde{x}) \cdot f(a+b-\tilde{x}) > f\left(\tfrac{a+b}{2}\right) \cdot f\left(\tfrac{a+b}{2}\right)$$

Because f is increasing, we know that $f(\tilde{x}) < f\left(\frac{a+b}{2}\right)$, and so

$$\cancel{f\left(\tfrac{a+b}{2}\right)} f(a+b-\tilde{x}) > f(\tilde{x}) \cdot f(a+b-\tilde{x}) > \cancel{f\left(\tfrac{a+b}{2}\right)} \cdot f\left(\tfrac{a+b}{2}\right)$$

$$\longrightarrow f(a+b-\tilde{x}) > f\left(\tfrac{a+b}{2}\right)$$

which contradicts the hypothesis that f is increasing. $\square$