Name: Patch

Quiz 2

right idea → +2
correct formula → +3

(1) Let $c(x)$ be the cubic function that interpolates value and the first derivative of $\sin x$ at $x = 0$ and $\frac{\pi}{2}$.

(a) (**3 Points**) Compute $c(x)$.

(b) (**2 Points**)From general Hermite error formula

$$p(x) - f(x) = \frac{f^{(2n+2)}(\zeta)}{(2n+2)!}\prod_{j=0}^{n}(x - x_j)^2,$$

show that

$$|c(x) - \sin x| \le \frac{1}{4!}\left(\frac{\pi}{4}\right)^4.$$

$\frac{1}{4!}\left(\frac{\pi}{2}\right)^2 \to +1$

$\frac{1}{4!}\left(\frac{\pi}{4}\right)^2 \to +2$

no justification: e.g., $\left|x^2\left(x-\frac{\pi}{2}\right)^2\right| \le \frac{\pi^2}{4}\cdot\frac{\pi^2}{4}$ → +1

(2) (**5 Points**) Let V be the space of functions on $[0, 1]$ with inner product

$$f \cdot g = f(0)g(0) + f'(1)g'(1).$$

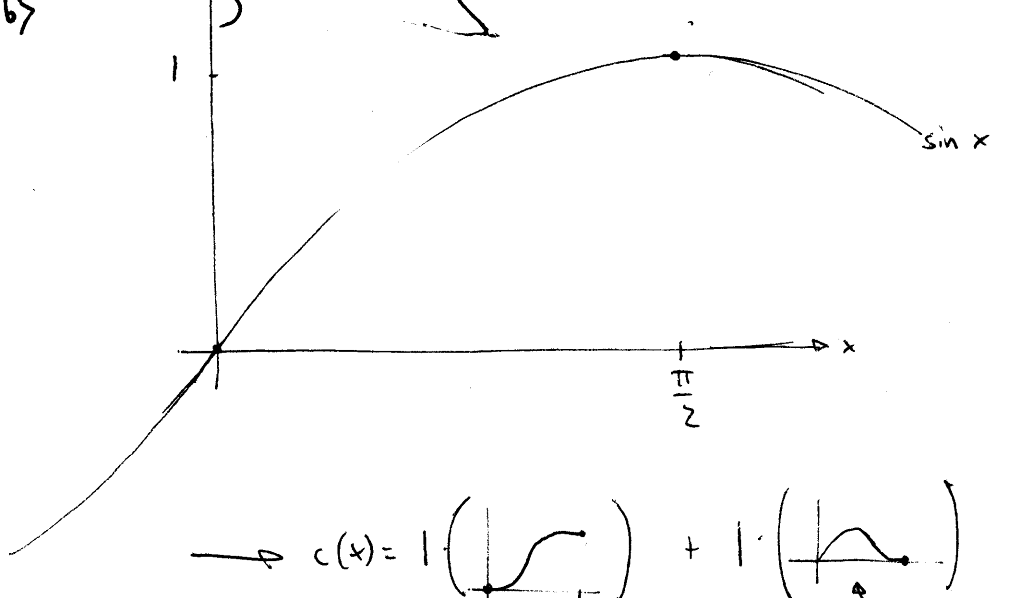
Compute the least square approximation to $b(x) = \cos\left(\frac{\pi x}{2}\right)$ in the form $a^*(x) = u(1 - x^2)$. (i.e. find u such that $|a^* - b|^2$ is minimum.)

$u = \frac{\langle a,b\rangle}{\langle a,a\rangle} \to +3$

$u = \frac{1+\pi}{1+4} \to +5$

$\langle e,a\rangle = 0 \to +1$
$u = \frac{\langle a,b\rangle}{\langle b,b\rangle} \to +1$
wrong, but better than nothing.

wrong calculus but w/ arithmetic errors → +4



$\to c(x) = 1\cdot(\ ) + 1\cdot(\ )$

$\left(x^2\cdot\frac{4}{\pi^2}\right)\cdot\left(1 + \gamma\cdot\left(x-\frac{\pi}{2}\right)\right)$

st. $2x\cdot\frac{4}{\pi^2}\left(1+\gamma\left(x-\frac{\pi}{2}\right)\right) + \left(x^2\cdot\frac{4}{\pi^2}\right)\cdot\gamma = 0$ @ $x = \frac{\pi}{2}$

$\to \gamma = -2\cdot\frac{2}{\pi}$

$\left(x-\frac{\pi}{2}\right)^2\cdot x\cdot\cancel{\gamma}\ \frac{4}{\pi^2}$

s.t. $\left[2\left(x-\frac{\pi}{2}\right)\cdot x + \left(x-\frac{\pi}{2}\right)^2\right]\cdot\gamma = 1$ @ $x=0$

$\to \gamma = \frac{4}{\pi^2}$

a)

$$\to \boxed{c(x) = \left(x^2\cdot\frac{4}{\pi^2}\right)\left(1 - \frac{4}{\pi}\left(x-\frac{\pi}{2}\right)\right) + \frac{4}{\pi^2}\left(x-\frac{\pi}{2}\right)^2\cdot x}$$

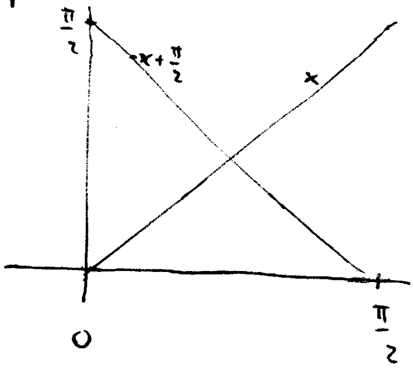
b) In our case, $f(x) = \sin x \longrightarrow |f^{[4]}(\xi)| \le 1 \quad \forall k, \xi$

Also in our case $n = 1$.

Using the general error formula, $|p(x) - f(x)| = \left|\frac{f^{[4]}(\xi)}{4!}x\left(x-\frac{\pi}{2}\right)\right| \le \frac{1}{4!}\underbrace{x^2\left(\frac{\pi}{2}-x\right)^2}_{\left|(x-0)^2\left(x-\frac{\pi}{2}\right)^2\right| \text{ on } \left[0,\frac{\pi}{2}\right]}$ & max

We will be done if we can show that $x^2 \cdot \left(\frac{\pi}{2} - x\right)^2 \leq \left(\frac{\pi}{4}\right)^4 \quad \forall x \in [0, \frac{\pi}{2}]$

Draw a picture:



Observe symmetry.

$\longrightarrow$ Obviously $x\left(\frac{\pi}{2} - x\right)$ has a max at $\frac{\pi}{4}$ of magnitude $\left(\frac{\pi}{4}\right)^2$. Can be rigorous about this if you want.

i.e.

$$x \cdot \left(\frac{\pi}{2} - x\right) \leq \left(\frac{\pi}{4}\right)^2 \quad \forall x \in [0, \tfrac{\pi}{2}]$$

The desired inequality can be obtained by squaring this one.

in total,

$$\left| p(x) - f(x) \right| \leq \frac{1}{4!}\left(\frac{\pi}{4}\right)^4 \quad \text{as desired.}$$

2.

$$u = \frac{\langle a, b \rangle}{\langle a, a \rangle} \quad \text{where} \quad a = (1 - x^2), \quad b = \cos\frac{\pi x}{2}$$

$$\langle a, b \rangle = \langle \underbrace{1 - x^2}_{-2x}, \underbrace{\cos\tfrac{\pi x}{2}}_{-\frac{\pi}{2}\sin\frac{\pi x}{2}} \rangle = 1 \cdot 1 + (-2)\cdot\left(-\tfrac{\pi}{2}\right) = 1 + \pi$$

$$\langle a, a \rangle = \langle \underbrace{1 - x^2}_{-2x}, \underbrace{1 - x^2}_{-2x} \rangle = 1 \cdot 1 + (-2)\cdot(-2) = 1 + 4$$

$\longrightarrow u = \dfrac{1 + \pi}{1 + 4}$ and the least squares approximation to $\cos\frac{\pi x}{2}$

is given by $\dfrac{1+\pi}{1+4}(1 - x^2)$.