Outline of the Error Formula Proofs

(For the formulas from the sheet titled "Summary of Polynomial Interpolation Error Formulas").

Case 1

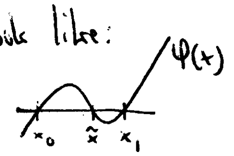
The error formula obviously holds when $\tilde{x} = x_0, x_1$, so pick $\tilde{x} \neq x_0, x_1$, (we'll show that it holds in this case too).

Let $\varphi(x) = f(x) - p(x) - \lambda(x - x_0)(x - x_1)$; clearly $\varphi(x) = 0$ at x_0 and x_1.

Pick λ so that $\varphi(x)$ also equals 0 at $\tilde{x}$ $\longrightarrow$ $\lambda = \dfrac{f(\tilde{x}) - p(\tilde{x})}{(\tilde{x} - x_0)(\tilde{x} - x_1)}$

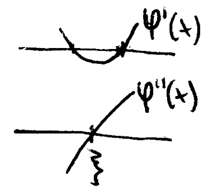
This causes $\varphi(x)$ to equal 0 at at least 3 distinct points (x_0, x_1, and $\tilde{x}$)

eg. $\varphi(x)$ might look like:



MVT $\longrightarrow$ $\varphi'(x) = 0$ at at least 2 distinct points.

MVT $\longrightarrow$ $\varphi''(x) = 0$ at at least 1 point, call it ξ.



That is,

$$0 = \varphi''(\xi) = f''(\xi) - \cancel{p''(\xi)} - \overbrace{\left(\frac{f(\tilde{x}) - p(\tilde{x})}{(\tilde{x} - x_0)(\tilde{x} - x_1)}\right)}^{\lambda} \cdot 2!$$

0 because p is a degree < 2 poly.

$2! = \dfrac{d^2}{dx^2}\left((x - x_0)(x - x_1) = x^2 + \cdots\right)$

Solve for $f(\tilde{x}) - p(\tilde{x})$

$$\longrightarrow f(\tilde{x}) - p(\tilde{x}) = \frac{f''(\xi)}{2!}(\tilde{x} - x_0)(\tilde{x} - x_1)$$

$\square$

If you understand the steps above, then it is easy to derive the error formula for interpolation through n points (this was done in lecture).

The derivations of the error formulas for Case 2 and Case 3 are similar...

turn

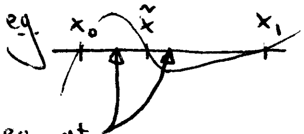
Case 2

The error formula holds when $\tilde{x} = x_0, x_1$, so pick $\tilde{x} \neq x_0, x_1$.

Let $\varphi(x) = f(x) - p(x) - \lambda (x - x_0)^2 (x - x_1)^2$

By construction $\varphi(x) = 0$ at x_0 and x_1; pick $\lambda = \dfrac{f(\tilde{x}) - p(\tilde{x})}{(\tilde{x} - x_0)^2 (\tilde{x} - x_1)^2}$ so that $\varphi(x)$ is also equal to 0 at $\tilde{x}$.

At this point, $\varphi(x)$ is zero at the distinct points x_0, x_1, and $\tilde{x}$, eg.



MVT $\longrightarrow$ $\varphi'(x)$ is zero at two points between the zeros of φ, eg. at

But we also have $\varphi' = 0$ at x_0 and x_1, by construction, giving in total at least 4 distinct points at which $\varphi' = 0$.

MVT $\longrightarrow$ φ'' has at least 3 zeros.

MVT $\longrightarrow$ φ''' has at least 2 zeros.

MVT $\longrightarrow$ φ'''' has a zero, call it ξ.

That is,

$$0 = \varphi^{[4]}(\xi) = f^{[4]}(\xi) - p^{[4]}(\xi) - \underbrace{\left(\frac{f(\tilde{x}) - p(\tilde{x})}{(\tilde{x} - x_0)^2 (\tilde{x} - x_1)^2}\right)}_{\lambda} \cdot 4!$$

0 because p is a degree < 4 poly.

$4! = \dfrac{d^4}{dx^4}\left((x - x_0)^2 (x - x_1)^2 = x^4 + \dots\right)$

Solve for $f(\tilde{x}) - p(\tilde{x})$

$$\longrightarrow f(\tilde{x}) - p(\tilde{x}) = \frac{f^{[4]}(\xi)}{4!} (\tilde{x} - x_0)^2 (\tilde{x} - x_1)^2$$

$\square$

Case 3

The error formula holds when $\hat{x} = x_0, x_1$ so pick $\hat{x} \neq x_0, x_1$.

Let $\varphi(x) = f(x) - p(x) - \lambda (x - x_0)^2 (x - x_1)$

By construction $\varphi(x) = 0$ at x_0 and x_1; pick $\lambda = \dfrac{f(\hat{x}) - p(\hat{x})}{(\hat{x} - x_0)^2 (\hat{x} - x_1)}$ so that $\varphi(x)$ is also zero at $\hat{x}$.

Thus $\varphi(x)$ is zero at the distinct points $x_0, x_1, \hat{x}$. (see Case 2 for a picture.)

MVT $\longrightarrow$ $\varphi'(x)$ is zero at two points between the zeros of φ.

But $\varphi' = 0$ at x_0 as well, giving a total of at least 3 distinct points at which $\varphi' = 0$.

MVT $\longrightarrow$ φ'' has at least 2 zeros.

MVT $\longrightarrow$ φ''' has a zero; call it ξ.

That is,

$$0 = \varphi^{[3]}(\xi) = f^{[3]}(\xi) - \cancel{p^{[3]}(\xi)} - \underbrace{\left(\frac{f(\hat{x}) - p(\hat{x})}{(\hat{x} - x_0)^2 (\hat{x} - x_1)}\right)}_{\lambda} \cdot 3!$$

0 because p is a degree < 3 poly.

$3! = \dfrac{d^3}{dx^3}\left((x - x_0)^2 (x - x_1) = x^3 + \cdots\right)$

Solve for $f(\hat{x}) - p(\hat{x})$

$$\longrightarrow f(\hat{x}) - p(\hat{x}) = \frac{f^{[3]}(\xi)}{3!} (\hat{x} - x_0)^2 (\hat{x} - x_1)$$

$\square$