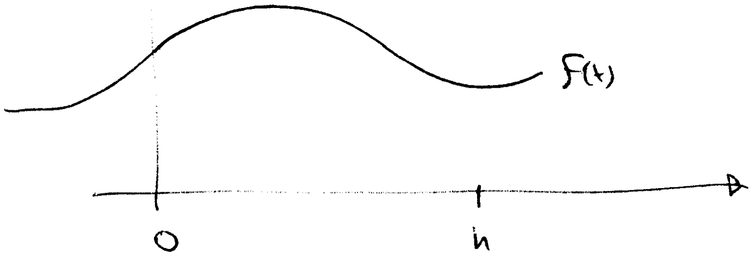
Midterm 2

1.

$$f(x) = \underbrace{f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f^{[3]}(0)}{3!}x^3}_{T(x)} + \underbrace{\frac{f^{[4]}(\xi)}{4!}x^4}_{\text{error term for taylor}}$$

(ξ between 0 and x.)

$$f(x) = \underbrace{f(0)\cdot + f'(0)\cdot + f(h)\cdot + f'(h)\cdot}_{h(x)} + \underbrace{\frac{f^{[4]}(\tilde{\xi})}{4!}(x-h)^2x^2}_{\text{error term for hermite}}$$

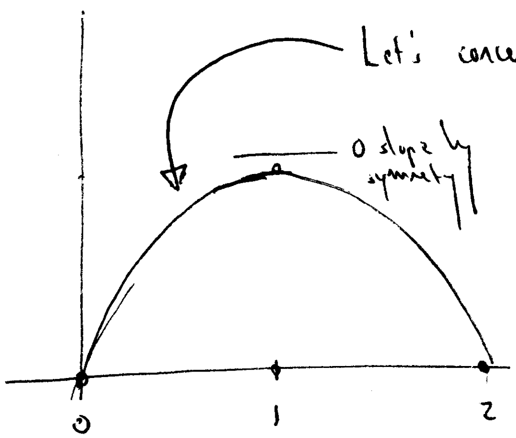


a). $$|T - f| = \left|\frac{f^{[4]}(\xi)}{4!}x^4\right| \le \frac{M}{4!}h^4 =: C$$

b) $$|h - f| = \left|\frac{f^{[4]}(\xi)}{4!}(x-h)^2x^2\right| \le \frac{M}{4!}\cdot\left(\frac{h}{2}\right)^4 = \frac{Mh^4}{4!}\cdot\frac{1}{16} < C \text{ as desired.}$$

↑ As on the quiz!

2.



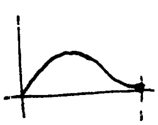
Let's concentrate on the first segment

$$h(x) = \cancel{f(0)}\cdot + \underbrace{f'(0)}_{\gamma}\cdot + \cancel{f(1)}\cdot + \cancel{f'(1)}\cdot$$

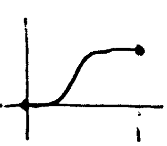
$$\rightarrow h(x) = \gamma\cdot + $$

- Find $\gamma = f'(0)$ so that $h''(0) = 0$
- compute $h''(1)$.

Recall:



$= x(x-1)^2 \xrightarrow{\frac{d}{dx}} (x-1)^2 + 2x(x-1)$



$= x^2(1 + \alpha(x-1)) \xrightarrow{\frac{d}{dx}} 2x(1+\alpha(x-1)) + \alpha x^2$

thus $\alpha = -2$

So

$$h(x) = \gamma\cdot x(x-1)^2 + x^2(3-2x)$$

$h(x) = \gamma(x^3 - 2x^2 + x) \quad - 2x^3 + 3x^2$

$h'(x) = \gamma(3x^2 - 4x + 1) \quad - 6x^2 + 6x$

$h''(x) = \gamma(6x - 4) \quad - 12x + 6$ (✡)

$\longrightarrow$ To have $h''(0) = 0$, we must have

$$\boxed{\gamma = f'(0) = \frac{3}{2} = 1.5}$$

Then easily we have

$$h''(1) = \frac{3}{2}(6-4) - 12 + 6$$

$$\longrightarrow \boxed{h''(1) = -3}$$

3. a) $u = \dfrac{\langle e_2, a\rangle}{\langle a, a\rangle}$ where $e_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $a = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ note $V := \text{span}(a)$

using the standard inner product

$\longrightarrow u = \dfrac{1}{4}$, so the projection of e_2 onto span a is $\dfrac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

This projection is the closest we can possibly get to e_2

$$\longrightarrow \min_{x \in V} |x - e_2| = \left|\frac{1}{4}a - e_2\right| = \left(\left\langle \begin{bmatrix}1/4\\-3/4\\1/4\\1/4\end{bmatrix}, \begin{bmatrix}1/4\\-3/4\\1/4\\1/4\end{bmatrix}\right\rangle\right)^{\frac{1}{2}} = \left(\frac{3}{16} + \frac{9}{16}\right)^{\frac{1}{2}} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}$$

b). $\underline{a}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \underline{a}_2 = \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \underline{a}_3 = \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$

First, find x_i s.t. $x_i \underline{a}_i$ is as close to $\underline{e}_2$ as possible

$$\begin{bmatrix}4 & & \\ & 4 & \\ & & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix} \longrightarrow$$

$\frac{1}{4}\underline{a}_1 + \frac{1}{4}\underline{a}_2 - \frac{1}{4}\underline{a}_3$ is the projection of $\underline{e}_2$ onto the space spanned by the $\underline{a}_i$'s. $(=: W)$

Thus

$$\min_{x \in V} |x - e_2| = \left(\left\langle \tfrac{1}{4}\underline{a}_1 + \tfrac{1}{4}\underline{a}_2 - \tfrac{1}{4}\underline{a}_3 - e_2,\ \tfrac{1}{4}\underline{a}_1 + \tfrac{1}{4}\underline{a}_2 - \tfrac{1}{4}\underline{a}_3 - e_2\right\rangle\right)^{\frac{1}{2}} = \frac{1}{2}$$

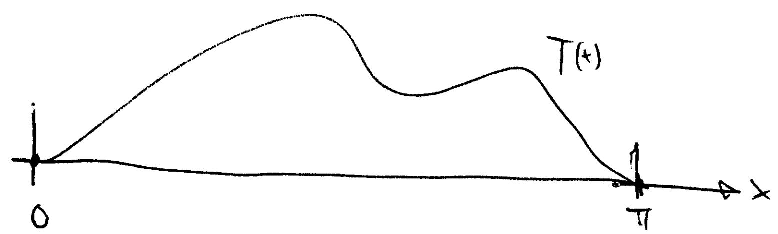
$$\begin{bmatrix}1/4\\-1/4\\1/4\\-1/4\end{bmatrix}$$

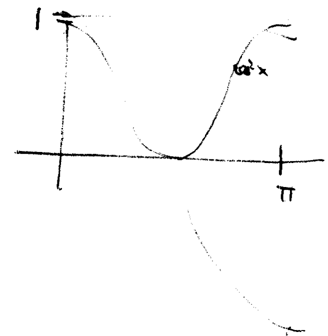
c). The error vector is orthogonal to W, and

4. ~~$E = \langle a_1, a_2 \rangle = \ldots$ if $f = a \cdot \sin x + \ldots$~~

~~$E = \int_0^{\pi} \left(a \cdot \frac{\cos^2 x}{2} - f(t)\, a \cdot \sin x\right) dt \ldots$~~

turn





a).

$T(t) = a \cdot \sin x$

Find a s.t.

$$E = \int_0^{\pi} \left(\frac{1}{2} a^2 \cos^2 x - f(x) \cdot a \sin x \right) dx$$ is minimized.

$$\longrightarrow E(a) = a^2 \cdot \frac{1}{2} \int_0^{\pi} \cos^2 x \, dx - a \cdot \int_0^{\pi} f(t) \cdot \sin x \, dx$$

$$\longrightarrow E(a) = \frac{\pi a^2}{4} - a \cdot \int_0^{\pi} f(t) \sin x \, dx$$

Thus $\frac{dE}{da} = 0 \longrightarrow \frac{\pi a}{2} = \int_0^{\pi} f(t) \sin x \, dx$

$$\longrightarrow \boxed{a = \frac{2}{\pi} \int_0^{\pi} f(t) \sin x \, dx}$$

If $f(t) = 1$ on $[0, \pi]$, then $\boxed{a = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx = \frac{-2}{\pi} \cos x \Big|_0^{\pi} = \frac{4}{\pi}}$

$$\longrightarrow \boxed{T\left(\frac{\pi}{2}\right) \approx \frac{4}{\pi} \cdot \sin \frac{\pi}{2} = \frac{4}{\pi}}$$

b). The exact T(x) has $T''(x) = -1$

$$\longrightarrow T'(x) = -x + C_1$$

$$\longrightarrow T(x) = \frac{-x^2}{2} + C_1 x + \cancel{C_2}^{\,0}$$

and $T(\pi) = \frac{-\pi^2}{2} + C_1 \pi = 0$

$$\longrightarrow C_1 = \frac{\pi}{2}$$

Thus exact solution given by $T(x) = \frac{-x^2}{2} + \frac{\pi}{2} x$

and $$\boxed{T\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 \left(1 - \frac{1}{2}\right) = \frac{\pi^2}{8}}$$