1. Let $f(x)$ be analytic in an interval containing $[0, h]$ and let $M$ be an upper bound on $|f''(x)|$ in $[0, h]$. Let $T(x)$ be the 3rd order Taylor polynomial of $f(x)$ about $x = 0$, and let $p(x)$ be the Hermite polynomial that interpolates $f(x)$ and $f'(x)$ at $x = 0$ and $x = h$.

a) Find an upper bound $C$ on $|T(x) - f(x)|$ in $[0, h]$ in terms of $M$, $h$.

b) Find an upper bound on $|p(x) - f(x)|$ in $[0, h]$ in terms of $M$, $h$, which is less than $C$. 
2. Let \( y(x) \) be the natural cubic spline which interpolates the 3 points \((0,0), (1,1), (2,0)\).
What are \( y(1) \) and \( y''(1) \)?

3. a) Let \( V \) be the subspace of \( \mathbb{R}^4 \) spanned by \( [1, 1, 0, 0] \). What is \( \min_{\text{in } V} |x - e_x| \), where \( e_x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \)?

b) Let \( W \) be the subspace of \( \mathbb{R}^4 \) spanned by \( \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \). What is \( \min_{\text{in } W} |x - e_x| \), where \( e_x \) is a vector orthogonal to any \( x \) in \( W \).
4. \(T(x)\) is the temperature field in a bar:

\[0 \leq x \leq a,\] subject to temperature zero at ends, \(T(0) = T(a) = 0\), and heat source fixed per unit length in \([0, a]\). Physically realized \(T(x)\) minimizes

\[E = \int_0^a \left\{ \frac{1}{2} (T'(x))^2 - f(x) T(x) \right\} \, dx.\] (11)

We seek approximation \(T(x) \approx a \sin x\), by substituting \(a \sin x\) for \(T(x)\) in (11) and determining the value of the constant \(a\) which minimizes \(E(a)\).

a) Find this value of \(a\), expressed as an integral. Evaluate for \(f(x) \equiv 1\) in \([0, a]\), and give the approximate value of \(T(\frac{\pi}{2})\) in this case.

b) For \(f(x) \equiv 1\), the exact \(T(x)\) has \(T''(x) \equiv -1\).
Compute the exact value of $f\left(\frac{\pi}{2}\right)$.

(Hint: It behooves ye to find quadratic $g(x)$ with $g''(x) = -1$ and $g(0) = g(\pi) = 0.$)