*Patch Kessler*

*Math128a*

# Trapezoidal Rule Error[1]

These notes show that the trapezoidal rule implented with subintervals of length $h$ has an error given by a series in *even* powers of $h$. Our main result is (6), which is an explicit formula for this series. Start by constructing a sequence of polynomials $p_k(t)$ for $k = 1, 2, 3, ...$ according to the following conditions

- $p_1(t) = -t$
- $\frac{dp_{k+1}(t)}{dt} = p_k(t)$
- if $k$ is odd, then $p_k(t)$ is an odd function of $t$.
- if $k > 1$ is odd, then $p_k(-1) = p_k(+1) = 0$.

Now consider some function $f(t)$ with first derivative $f^{[1]}$ that exists and that can be integrated on $[-1, 1]$. Integrating by parts, we see that

$$\int_{-1}^{1} f(t)dt - (f(-1) + f(1)) = -\int_{-1}^{1} tf^{[1]}(t)dt = \int_{-1}^{1} p_1(t)f^{[1]}(t)dt \tag{1}$$

If $f^{[2l]}$ exists and can be integrated on $[-1, 1]$ for some positive integer $l$, then repeated integration by parts transforms the right hand side of the above equation as follows

$$\begin{aligned}
\int_{-1}^{1} p_1(t)f^{[1]}(t)dt &= \left(p_2 f^{[1]}\right)_{-1}^{1} - \int_{-1}^{1} p_2(t)f^{[2]}(t)dt \\
&= \left(p_2 f^{[1]} - p_3 f^{[2]}\right)_{-1}^{1} + \int_{-1}^{1} p_3(t)f^{[3]}(t)dt \\
&\vdots \\
&= \left(p_2 f^{[1]} - p_3 f^{[2]} + p_4 f^{[3]} - p_5 f^{[4]} + \cdots + p_{2l} f^{[2l-1]}\right)_{-1}^{1} - \int_{-1}^{1} p_{2l}(t)f^{[2l]}(t)dt \\
&= \left(p_2 f^{[1]} + p_4 f^{[3]} + \cdots + p_{2l} f^{[2l-1]}\right)_{-1}^{1} - \int_{-1}^{1} p_{2l}(t)f^{[2l]}(t)dt \qquad (2)
\end{aligned}$$

where to cancel terms in the last line, we've used the fact that $p_k(-1) = p_k(+1) = 0$ when $k > 1$ is odd. Notice that $f(-1) + f(1)$ is the basic trapezoidal rule approximation of $\int_{-1}^{1} f(t)dt$. The idea now is that we scale these results and apply them to each segment of the composite trapezoidal rule. Let $g(x)$ be a function so that for some positive integer $l$, $g^{[2l]}$ exists and can be integrated on $[a, b]$. Let $h = m^{-1}(b - a)$ for some positive integer $m$, and let $T(h)$ denote the composite trapezoidal rule approximation of $\int_a^b g(x)dx$ using segments of length $h$. The error associated with this approximation is given by

$$\int_a^b g(x)dx - T(h) = \sum_{k=0}^{m-1} \left( \int_{x_k}^{x_{k+1}} g(x)dx - \frac{h}{2}(g(x_k) + g(x_{k+1})) \right) \tag{3}$$

where $x_k = a + kh$. Make $x \in [x_k, x_{k+1}]$ a function of $t$ according to $x(t) = t\frac{h}{2} + \frac{x_k + x_{k+1}}{2}$, and let $f_k(t)$ denote the resulting composite function $g(x(t))$. This allows us to write

$$\begin{aligned}
\int_a^b g(x)dx - T(h) &= \frac{h}{2} \sum_{k=0}^{m-1} \left( \int_{-1}^{1} f_k(t)dt - (f_k(-1) + f_k(1)) \right) \\
&= \frac{h}{2} \sum_{k=0}^{m-1} \left( \left(p_2 f_k^{[1]} + p_4 f_k^{[3]} + \cdots + p_{2l} f_k^{[2l-1]}\right)_{-1}^{1} - \int_{-1}^{1} p_{2l}(t) f_k^{[2l]}(t)dt \right) \qquad (4)
\end{aligned}$$

[1]My derivation is adapted from *An Introduction to Numerical Analysis* by Suli and Mayers.

where (1) and (2) have been used to obtain the second line. Notice that $p_k(t)$ is an even function of $t$ when $k$ is even, that $f_k^{[n]}(+1) = f_{k+1}^{[n]}(-1)$ for $k = 0, 1, \ldots, m-2$ and for any $n$, and that $f_k^{[n]}(t) = \left(\frac{h}{2}\right)^n g^{[n]}(x(t))$ where of course we're using the $x(t)$ corresponding to the interval $[x_k, x_{k+1}]$. Then for $n = 1, 2, 3, \ldots$ it follows that

$$\begin{aligned}\sum_{k=0}^{m-1} \left(p_{2n} f_k^{[2n-1]}\right)_{-1}^{1} &= p_{2n}(1)\left(f_{m-1}^{[2n-1]}(1) - f_0^{[2n-1]}(-1)\right) \\ &= p_{2n}(1)\left(\frac{h}{2}\right)^{2n-1}\left(g^{[2n-1]}(b) - g^{[2n-1]}(a)\right) \end{aligned} \tag{5}$$

which allows us to write (3) as

$$\begin{aligned}\int_a^b g(x)dx - T(h) = h^2 &\frac{p_2(1)}{2^2}\left(g^{[1]}(b) - g^{[1]}(a)\right) \\ &+ h^4 \frac{p_4(1)}{2^4}\left(g^{[3]}(b) - g^{[3]}(a)\right) \\ &+ h^6 \frac{p_6(1)}{2^6}\left(g^{[5]}(b) - g^{[5]}(a)\right) \\ &\vdots \\ &+ h^{2l} \frac{p_{2l}(1)}{2^{2l}}\left(g^{[2l-1]}(b) - g^{[2l-1]}(a)\right) \\ &- \frac{h^{2l}}{2^{2l}} \sum_{k=0}^{m-1} \int_{x_k}^{x_{k+1}} p_{2l}(t_k(x)) g^{[2l]}(x) dx \end{aligned} \tag{6}$$

where $t_k(x) = \frac{2x}{h} - \frac{x_k + x_{k+1}}{h}$. Equation (6) is our main result, however our work is far from done. To fully appreciate this expansion, we need to develop an intuition for the numbers $p_{2k}(1)$. The first few polynomials $p_k(t)$ are given by

$$\begin{aligned} p_1(t) &= -1 \\ p_2(t) &= \frac{-t^2}{2} + \frac{1}{6} \\ p_3(t) &= \frac{-t^3}{6} + \frac{t}{6} \\ p_4(t) &= \frac{-t^4}{24} + \frac{t^2}{12} - \frac{7}{360} \end{aligned} \tag{7}$$

A little work reveals that generally, for k=1,2,3,...

$$\begin{aligned} p_{2k}(t) &= c_1 \frac{t^{2k}}{(2k)!} + c_3 \frac{t^{2k-2}}{(2k-2)!} + c_5 \frac{t^{2k-4}}{(2k-4)!} + \cdots + c_{2k-3} \frac{t^4}{4!} + c_{2k-1} \frac{t^2}{2!} + c_{2k+1} \frac{t^0}{0!} \\ p_{2k-1}(t) &= c_1 \frac{t^{2k-1}}{(2k-1)!} + c_3 \frac{t^{2k-3}}{(2k-3)!} + c_5 \frac{t^{2k-5}}{(2k-5)!} + \cdots + c_{2k-3} \frac{t^3}{3!} + c_{2k-1} \frac{t^1}{1!} \end{aligned} \tag{8}$$

To calculate the coefficients $c_1$, $c_3$, note that the identity $p_{2k+1}(1) = 0$ gives us

$$\frac{c_1}{(2k-1)!} + \frac{c_3}{(2k-3)!} + \frac{c_5}{(2k-5)!} + \cdots + \frac{c_{2k-5}}{5!} + \frac{c_{2k-3}}{3!} + \frac{c_{2k-1}}{1!} = 0. \tag{9}$$

which can be used to find $c_{2k-1}$ in terms of earlier coefficients, with $c_1 = -1$ used as a starting point. Using code pasted to the end of these notes I find that the initial $c_k$'s are given by

| | numerator | denominator | quotient |
|---|---|---|---|
| $c_1$ | -1 | 1 | -1.00000000000000 |
| $c_3$ | 1 | 6 | 0.16666666666667 |
| $c_5$ | -7 | 360 | -0.01944444444444 |
| $c_7$ | 31 | 15120 | 0.00205026455026 |
| $c_9$ | -127 | 604800 | -0.00020998677249 |
| $c_{11}$ | 73 | 3421440 | 0.00002133604564 |
| $c_{13}$ | -1414477 | 653837184000 | -0.00000216334744 |
| $c_{15}$ | 8191 | 37362124800 | 0.00000021923271 |
| $c_{17}$ | -16931177 | 762187345920000 | -0.00000002221393 |

and that the initial $p_k$'s are given by

| | numerator | denominator | quotient |
|---|---|---|---|
| $p_2(1)$ | -1 | 3 | -0.33333333333333 |
| $p_4(1)$ | 1 | 45 | 0.02222222222222 |
| $p_6(1)$ | -2 | 945 | -0.00211640211640 |
| $p_8(1)$ | 1 | 4725 | 0.00021164021164 |
| $p_{10}(1)$ | -2 | 93555 | -0.00002137779916 |
| $p_{12}(1)$ | 1382 | 638512875 | 0.00000216440428 |
| $p_{14}(1)$ | -4 | 18243225 | -0.00000021925948 |
| $p_{16}(1)$ | 3617 | 162820783125 | 0.00000002221461 |

The first few values $p_k(1)$ are approximately given by the alternating geometric sequence $p_{k+2}(1) = -\gamma p_k(1)$ with $\gamma = 0.1$, however the magnitude of $\gamma$ seems to decrease as $k$ gets bigger. If anyone has further information about this, please let me know and I'll include it in these notes.

```
function [c,p]=traperror(n)
%compute the coefficients p_2, p_4, ... , p_{2n} associated with the
%trapezoidal rule error expansion.  Also compute c_1, c_3, ...

f=1;    g=2;
cn=-1;  cd=1;
for k=1:n
    f=f.*g.*(g+1);
    [newcn,newcd]=fracsum(-1*cn,cd.*f);
    cn=[cn;newcn];  cd=[cd;newcd];
    f=[f;1];
    g=[2+g(1);g];
    [newpn,newpd]=fracsum((g-1).*cn,f.*cd);
    pn(k,1)=newpn;    pd(k,1)=newpd;
end
c=int64([cn cd]);
p=int64([pn pd]);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [nsum,dsum]=fracsum(n,d)

div=gcd(round(n),round(d));
n=round(n./div);
d=round(d./div);
dsum=1;
for k=1:length(d)
    dsum=lcm(dsum,d(k));
end
nsum=dsum*sum(n./d);
div=gcd(round(nsum),round(dsum));
nsum=nsum/div;
```