Significant Digits

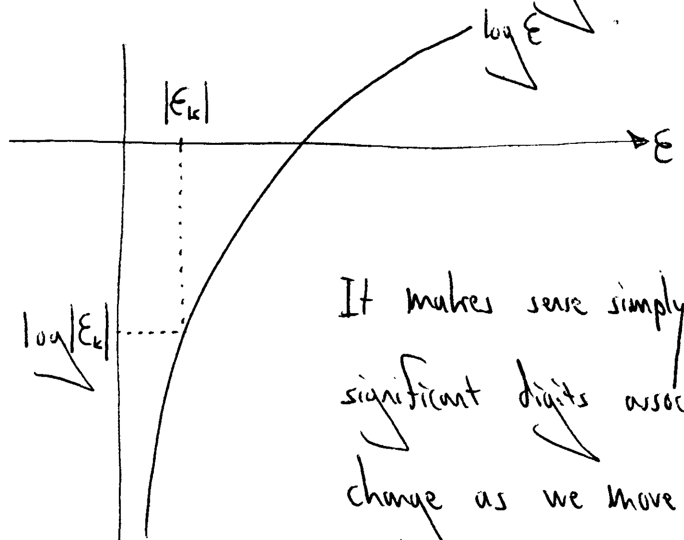
Let $x_k \longrightarrow \alpha$

Agree that "x_k is to within m significant digits of α" iff $|x_k - \alpha| < 10^{-m}$.

Let $\varepsilon_k = x_k - \alpha$, and take logarithms

$$|x_k - \alpha| < 10^{-m} \longleftrightarrow |\varepsilon_k| < 10^{-m} \longleftrightarrow \log|\varepsilon_k| < -m$$

Note that for small $|\varepsilon_k|$, $\log|\varepsilon_k|$ is a big negative number.



It makes sense simply to call $-\log|\varepsilon_k|$ the number of significant digits associated with x_k. How does this number change as we move to the next iterate, x_{k+1}?

For ε_k small, $\dfrac{\varepsilon_{k+1}}{\varepsilon_k^s} \approx C \longrightarrow \left(-\log|\varepsilon_{k+1}|\right) = s\left(-\log|\varepsilon_k|\right) + \left(-\log|C|\right)$ $\quad (\star)$

In words: $\begin{pmatrix}\text{new \# of}\\ \text{sig. digits}\end{pmatrix} = s \cdot \begin{pmatrix}\text{old \# of}\\ \text{sig. digits}\end{pmatrix} + \begin{pmatrix}\text{a constant}\end{pmatrix}$

Equation $(\star)$ is what you need to work out problems.

Important Cases

- When $s=1$ (called linear convergence), the only increase in the number of significant digits comes from the constant $-\log|C|$. For instance let $x_{k+1}=g(x_k)$ and suppose that $g(\alpha)=\alpha$ and $g'(\alpha)\neq 0$.

$$x_{k+1}=g(x_k)=g(\alpha+\varepsilon_k)=\overset{\alpha}{\cancel{g(\alpha)}}+g'(\xi)\cdot\varepsilon_k \quad \text{for some } \xi \text{ between } \alpha \text{ and } x_k.$$

$$\longrightarrow \frac{\varepsilon_{k+1}}{\varepsilon_k}=g'(\xi)$$

Supposing x_k is close to α, it follows that ξ (between x_k and α) is also close to α, and so $g'(\xi)\approx g'(\alpha)$ (assume g' continuous). Thus C for this example is $g'(\alpha)$. For the number of significant digits to increase, we need $-\log|C=g'(\alpha)|>0$, which happens if $|g'(\alpha)|<1$, which is a familiar sufficient condition for convergence at α.

- When $s>1$, observe that

$$\begin{pmatrix}\text{increase in \#}\\ \text{of significant}\\ \text{digits after 1}\\ \text{iteration}\end{pmatrix} = (s-1)\cdot\begin{pmatrix}\text{old \# of}\\ \text{significant}\\ \text{digits}\end{pmatrix} + \begin{pmatrix}\text{a constant}\end{pmatrix}$$

As $\varepsilon_k\longrightarrow 0$, $-\log|\varepsilon_k|$ gets huge compared to the unchanging $-\log|C|$. For this reason, the constant term is often neglected, giving

$$\begin{pmatrix}\text{new \# of}\\ \text{sig. digits}\end{pmatrix}\approx s\cdot\begin{pmatrix}\text{old \# of}\\ \text{sig. digits}\end{pmatrix}$$