Rate of Convergence of a Sequence.

A sequence \( x_k \) is said to converge to \( \alpha \) with order \( s \) if \( \lim \frac{E_{k+1}}{E_k^s} = c \), where \( c \) is a nonzero constant, (and of course \( E_k = x_k - \alpha \)). Most theoretical problems in Math 128a involve showing that \( s = 1 \) (called linear convergence) or that \( s = 2 \) (called quadratic convergence). Note that

\[
\log E_{k+1} = s \log E_k + \log c
\]

and so a practical method of finding \( s \) and \( c \) is to run code on a computer, generating error values \( E_k \), and to then plot \( \log E_{k+1} \) versus \( \log E_k \). The data points should fall roughly on a line, the slope of which is \( s \), and the \( y \)-intercept \( s \) of which is \( \log c \).

It is worth noting that in practice, a sequence undergoing quadratic convergence gets to within numerical precision (\( \approx 10^{-16} \)) of its limit after only a handful of iterations. Although for theoretical problems \( s \) is normally 1 or 2, we can show that for the secant method, \( s = \frac{1 + \sqrt{5}}{2} \approx 1.61803 \ldots \)
Recall that the Secant Method generates a new iterate $x_{k+2}$ given two previous iterates $x_k$ and $x_{k+1}$ as illustrated below.

\[ F(x_k) = F'(0) \cdot x_k + \frac{F''(0)}{2} \cdot x_k^2 + O(x_k^3) \]

\[ F(x_{k+1}) = F'(0) \cdot x_{k+1} + \frac{F''(0)}{2} \cdot x_{k+1}^2 + O(x_{k+1}^3) \]

Thus,

\[ x_{k+1} \cdot F(x_k) - x_k \cdot F(x_{k+1}) = \frac{F''(0)}{2} \cdot (x_{k+1} \cdot x_k^2 - x_k \cdot x_{k+1}^2) + O(x_k^3) \]

\[ = \frac{F''(0)}{2} \cdot x_k \cdot x_{k+1} \cdot (x_k - x_{k+1}) + O(x_k^3) \]

Also,

\[ F(x_k) - F(x_{k+1}) = F'(0) \cdot (x_k - x_{k+1}) + O(x_k^2) \]

Thus (3) can be written as

\[ x_{k+2} = F'(0) \cdot (x_k - x_{k+1}) \left( 1 + O(x_k) \right) = F'(0) \cdot (x_k - x_{k+1}) \left( \frac{1}{2} \frac{F''(0)}{F'(0)} x_k x_{k+1} + O(x_k^3) \right) \]

Recall from a problem set that $\frac{1}{1 - x} = 1 - x + x^2 \ldots$ and so

\[ x_{k+2} = \frac{1}{2} \frac{F''(0)}{F'(0)} x_k x_{k+1} + O(x_k^3) \]
when the $x$'s are close enough to the root $(0)$ we have

$$x_{k+2} = M x_{k+1} x_k$$

where $M = \frac{1}{2} \sum_n a(0) s_n$ is assumed nonzero (or we'd have to look at even higher order terms!). Define $y_k = M x_k$ and note that

$$x_{k+2} = M y_{k+1} y_k$$

$$y_{k+1} = M y_k$$

And so $y_{k+2} = y_{k+1} y_k \rightarrow z_{k+2} = z_{k+1} + z_k$, where $z_k = \log y_k$.

Notice that

$$\begin{pmatrix} z_{k+1} \\ z_{k+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_k \\ y_k \end{pmatrix}$$

which gives the nice expression $y_k = A^k y_0$. For any $y_0$, a short computation reveals that $A$ has eigenvalues $\lambda_1 = \frac{1 + \sqrt{5}}{2}$ and $\lambda_2 = \frac{1 - \sqrt{5}}{2}$, and corresponding eigenvectors

$y_1 = \begin{bmatrix} \lambda_1 - 1 \\ 1 \end{bmatrix}$ and $y_2 = \begin{bmatrix} \lambda_2 - 1 \\ 1 \end{bmatrix}$, which obviously span $\mathbb{R}^2$. Writing $y_0 = \alpha y_1 + \beta y_2$, we obtain $y_k = A^k y_0 = \alpha A^k y_1 + \beta A^k y_2 = \alpha \lambda_1^k y_1 + \beta \lambda_2^k y_2$. Note that $|\lambda_2| < 1$ and so $\lambda_2^k \to 0$ as $k \to \infty$, thus for large $k$, $y_k \approx \alpha \lambda_1^k y_1$. Taking the $2n$th component we obtain $z_{k+2} \approx \alpha \lambda_1^k$. A sufficient condition for $z_k \to \infty$ is $\alpha < 0$; for convenience we take $\alpha = -1$, (it is easy to see that $\alpha$ doesn't affect the following argument).

$$z_{k+2} = \frac{y_k x_{k+1}}{M} \rightarrow x_{k+2} = M e^{-\lambda_1^k}$$

\[ \text{e}^\theta \]
Remember that our original task was to find $s$ so that $\frac{E_{k+1}}{E_k} = R \to$ a non-zero constant.

$$R = \frac{E_{k+1}}{E_k} = \frac{x_{k+1}}{x_k} = \frac{Me^{-\lambda_{k+1}}}{Me^{-s\lambda_k}} = e^{s\lambda_k - \lambda_{k+1}} = e^{\lambda_k(s - \lambda_1)}$$

If $s > \lambda_1$, then clearly $R$ goes to either 0 or infinity. It must be that

$$s = \lambda_1 = \frac{1 + \sqrt{5}}{2}$$