__Linear__ ODE − any linear combination of solutions is a solution

1. __First order__ $y' + \underbrace{p(x)}_{\text{given}} y = 0$

$$\int \frac{dy}{y} = -\int p(x)\,dx \quad \text{or} \quad \ln y = -\underbrace{P(x)}_{\substack{\text{antiderivative}\\ \text{of } p(x)}} + \text{const}$$

$$y = \text{const}\, e^{-P(x)}$$

__Inhomogeneous__ ODE $y' + p(x) y = \underbrace{f(x)}_{\text{given}}$

$$e^{P}(y' + P' y) = e^{P} f$$

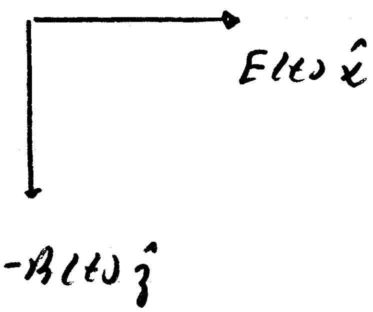
$$(e^{P} y)' = e^{P} f$$

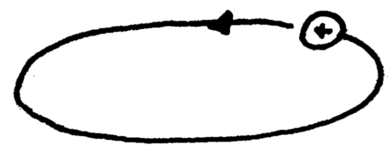
$$\int_{x_0}^{x} (e^{P} y)'\,dt = \int_{x_0}^{x} e^{P} f\,dt$$

$$[e^{P} y]_{x_0}^{x} = e^{P(x)} y(x) - e^{P(x_0)} \underbrace{y(x_0)}_{\text{assume } = 0} = \int_{x_0}^{x} e^{P(t)} f(t)\,dt$$

$$y(x) = y(x_0)\, e^{-P(x)} + \int_{x_0}^{x} e^{P(t) - P(x)} f(t)\, dt$$

ex) Charged particle in magnetic and electric fields





Velocity in x, y plane

$$u(t)\,\hat{x} + v(t)\,\hat{y}$$

$u(t)$, $v(t)$ satisfy ODEs

$$\dot{u} = -B(t)\, v + E(t)$$

$$\dot{v} = B(t)\, u$$

Define complex $z(t) = u(t) + i\, v(t)$

$$\dot{z} = \dot{u} + i\dot{v} = B(t)(-v + i u) + E(t)$$

$$= iB(t)(u + iv) + E(t)$$

$$= i\, B(t)\, z + E(t)$$

$$\Rightarrow \quad \mathring{z} - iB(t)z = E(t)$$

Sol'tns of homogeneous ODE ($E(t) \equiv 0$) prop to $e^{i\theta(t)}$, $\mathring{\theta}(t) = B(t)$. Write ODE as

$$(e^{-i\theta} z)^{\circ} = e^{-i\theta} E.$$

<u>IVP</u> Suppose $z(0) = z_0$ given. Take $\theta(0) = 0$

$$e^{-i\theta(t)} z(t) - z_0 = \int_0^t e^{-i\theta(t')} E(t')\,dt'$$

$$z(t) = z_0 e^{i\theta(t)} + \int_0^t e^{i(\theta(t) - \theta(t'))} E(t')\,dt'$$

Take $B(t) = \Omega = \text{const}$, and $E(t) = \cos \Omega t$, and $z_0 = 0$. Then

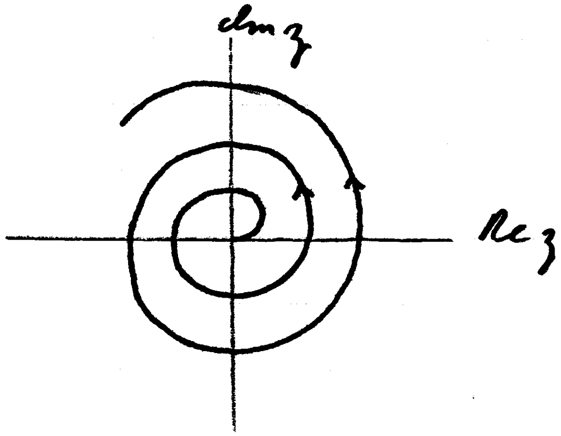
$$z(t) = \int_0^t e^{i\Omega(t-t')} \cos \Omega t'\,dt'$$

$$= e^{i\Omega t} \int_0^t e^{-i\Omega t'} \left\{ \tfrac{1}{2} e^{i\Omega t'} + \tfrac{1}{2} e^{-i\Omega t'} \right\} dt'$$

$$= \tfrac{1}{2} e^{i\Omega t} \int_0^t \left\{ 1 + e^{-2i\Omega t'} \right\} dt'$$

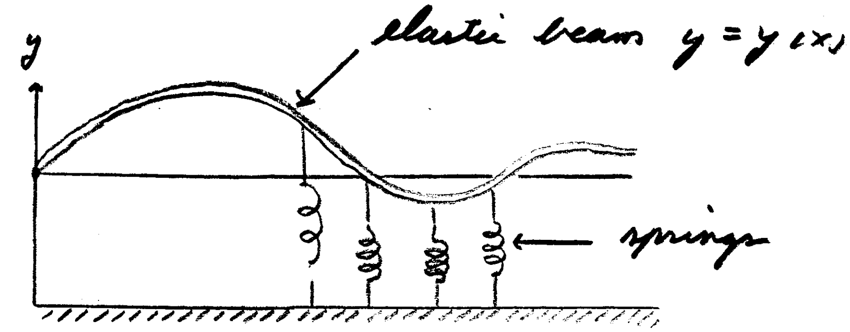
$$= \tfrac{1}{2} e^{i\Omega t} \left\{ t - \frac{1}{2i\Omega} (e^{-2i\Omega t} - 1) \right\} \Rightarrow$$

$$z(t) = \underbrace{\frac{t}{2} e^{i\Omega t}} + \frac{\sin \Omega t}{2\Omega}$$



2 __Exponential solutions of const coef linear ODEs__

__example__



$$y'''' + y = 0 \text{ in } x > 0.$$

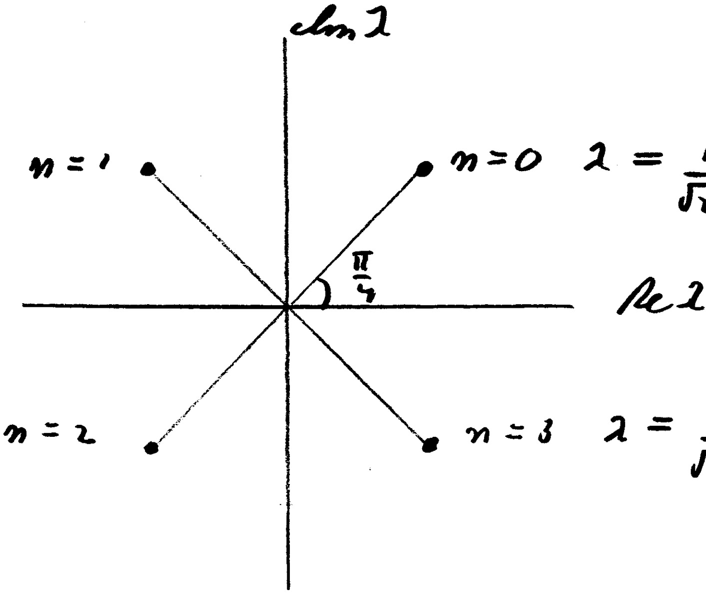
Seek $y(x) = e^{\lambda x} \Rightarrow \lambda^4 + 1 = 0$ or

$\lambda^4 = -1$ (4th roots of -1). Write $\lambda = re^{i\vartheta}$,

$-1 = e^{i\pi}$ so $r^4 e^{4i\vartheta} = e^{i\pi} \Rightarrow r^4 e^{i(4\vartheta - \pi)} = 1$.

$|\ \ | \Rightarrow r = 1$, and $4\vartheta - \pi = 2\pi n$ or

$$\nu = \frac{\pi}{4} + \frac{\pi}{2} n$$



$\lambda = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ $n = 1$

$n = 0$ $\lambda = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$\lambda = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$ $n = 2$

$n = 3$ $\lambda = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$n = 4$ repeats $n = 0$
" 5 " $n = 1$ etc.

Hence, "elementary solutions"

$$e^{\pm\frac{x}{\sqrt{2}} \pm i\frac{x}{\sqrt{2}}}$$

<u>alternative</u> set by <u>linear combinations</u>

$$\frac{1}{2} e^{\frac{x}{\sqrt{2}} + i\frac{x}{\sqrt{2}}} + \frac{1}{2} e^{\frac{x}{\sqrt{2}} - i\frac{x}{\sqrt{2}}} =$$

$$e^{\frac{x}{\sqrt{2}}} \frac{1}{2}\left\{ e^{i\frac{x}{\sqrt{2}}} + e^{-i\frac{x}{\sqrt{2}}} \right\} = e^{\frac{x}{\sqrt{2}}} \cos\frac{x}{\sqrt{2}} .$$

Similiarly get complete set

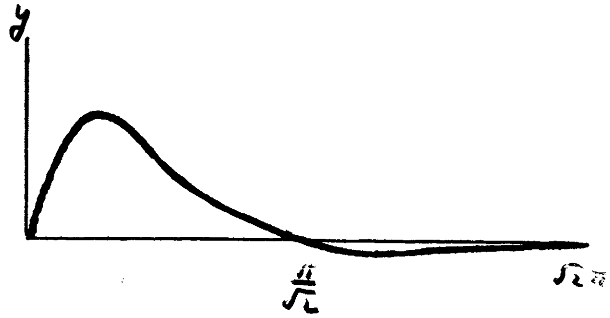
$$\left.\begin{matrix} e^{\frac{x}{\sqrt{2}}}\cos\frac{x}{\sqrt{2}}, & e^{-\frac{x}{\sqrt{2}}}\cos\frac{x}{\sqrt{2}} \\[2ex] e^{\frac{x}{\sqrt{2}}}\sin\frac{x}{\sqrt{2}}, & e^{-\frac{x}{\sqrt{2}}}\sin\frac{x}{\sqrt{2}} \end{matrix}\right]$$

<u>general</u> soltn = lin comb of these

<u>BVP</u> $y(0) = 0, \quad y'(0) = 1, \quad \underbrace{y \to 0 \text{ as } x \to \infty}.$

y is linear comb of <u>decaying</u> solutions ie

$$y = e^{-\frac{x}{\sqrt{2}}}\left\{ \underset{\substack{\uparrow \\ y(0)=0 \\ \Rightarrow a=0}}{a} \cos\frac{x}{\sqrt{2}} + \underset{\substack{\uparrow \\ y'(0)=1 \\ \Rightarrow b=\sqrt{2}}}{b} \sin\frac{x}{\sqrt{2}} \right\} = \sqrt{2}\, e^{-\frac{x}{\sqrt{2}}} \sin\frac{x}{\sqrt{2}}$$



Systems (Hee-Hee-Hoo)

$x(t) \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ satisfies $\dot{x} = A x$, $A = \begin{bmatrix} -\varepsilon & -1 \\ 1 & -\varepsilon \end{bmatrix}$

Exponential solutions $x = \underset{\substack{\uparrow \\ \text{const vector}}}{v} e^{\lambda t}$ substitute

into $\dot{x} = A x \Rightarrow \lambda v = A v \Longleftrightarrow \underbrace{(A - \lambda I) v = 0}$

$\det(A - \lambda I) =$ algebraic e-value

$\det \begin{bmatrix} -\varepsilon - \lambda & -1 \\ 1 & -\varepsilon - \lambda \end{bmatrix} =$ problem

$(\varepsilon + \lambda)^2 + 1 = 0 \Rightarrow \lambda = -\varepsilon + j$ or $-\varepsilon - j$.

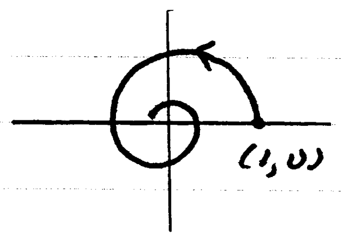
For $\lambda = -\varepsilon + j$, $A - \lambda I = \begin{bmatrix} -j & -1 \\ 1 & -j \end{bmatrix}$ and

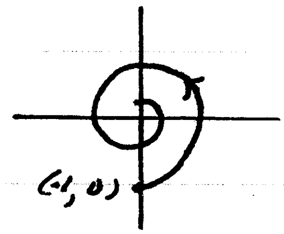
e-vector v is $v = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Get exp soltn'

$$x = \begin{bmatrix} 1 \\ -j \end{bmatrix} e^{(-\varepsilon + j) t}.$$

Real soltn's by taking Re and Im parts

$$\mathrm{Re}\, x = \begin{bmatrix} e^{-\varepsilon t}\cos t \\ e^{-\varepsilon t}\sin t \end{bmatrix}, \quad \mathrm{Im}\, x = \begin{bmatrix} e^{-\varepsilon t}\sin t \\ -e^{-\varepsilon t}\cos t \end{bmatrix}$$

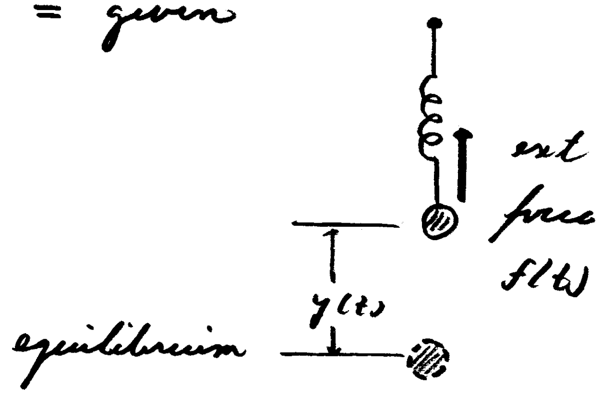




Gen real soltn is lin comb of these.

3) **Inhomogeneous** const coef, linear ODE's

example $\ddot{y} + y = f(t)$ = given



Equiv system
$$\left.\begin{aligned} \dot{y} &\equiv v \\ \dot{v} &= -y + f \end{aligned}\right] \iff$$

$$\underbrace{\begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix}}_{\dot{Y}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\equiv A} \underbrace{\begin{bmatrix} y \\ v \end{bmatrix}}_{\equiv Y} + \underbrace{\begin{bmatrix} 0 \\ f \end{bmatrix}}_{\equiv F}$$

or

$$\dot{Y} = AY + F \qquad (1)$$

1) Homog ODE's $\dot{Y} = AY$ have elementary solutions

$Y_1 \equiv \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$, $Y_2 \equiv \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$. Form <u>soltn matrix</u>

$$S = [Y_1\ Y_2] = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$ Rot by angle $-t$.

S satisfies

$$\dot{S} = [\dot{Y}_1 \;\; \dot{Y}_2] = [AY_1 \;\; AY_2] = A[Y_1 \;\; Y_2] = AS.$$

Seek solutions of inhomogeneous ODE's (F 70)

in form $Y = S(t)\underbrace{C(t)}_{\text{column vector function of } t}$

Compute

$$\dot{Y} = S\dot{C} + \dot{S}C = S\dot{C} + \underbrace{ASC}_{AY} = AY + \underbrace{S\dot{C}}_{\text{must be } F}$$

So $S\dot{C} = F$ and $\dot{C} = S^{-1}F$, and

$$C(t) = C(0) + \int_0^t S^{-1}(t')\,F(t')\,dt,$$

and finally

$$Y(t) = S(t)C(t) = S(t)\underbrace{C(0)}\; + \int_0^t \underbrace{S(t)\,S^{-1}(t')}\,F(t')\,dt'$$

$y(0) = v(0) = \dot{y}(0) = 0 \Leftrightarrow$ take 0 $\quad$ rotation by

$t' - t$ $\quad$ (2)

$$S(t)\,S^{-1}(t')\,F(t') = \begin{bmatrix} \cos(t'-t) & -\sin \\ \sin & \cos \end{bmatrix} \begin{bmatrix} 0 \\ f(t') \end{bmatrix}$$

1st component of (2)

$$y(t) = -\int_0^t \sin(t'-t)\, f(t')\, dt'$$

or

$$\boxed{y(t) = \int_0^t \sin(t-t')\, f(t')\, dt'}$$