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Planar Chain Dynamics

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1 Overview

We present the equations of motion for a chain of n point masses in a plane. The i^{th} mass m_i is located by the position vector $\mathbf{x}_i$. We impose the constraints $\|\mathbf{x}_1\| = l$, and $\|\mathbf{x}_i - \mathbf{x}_{i-1}\| = l$ for $i = 2, ..., n$, and we define the vectors $\{\mathbf{e}_i\}$ so that $l\mathbf{e}_1 = \mathbf{x}_1$, and $l\mathbf{e}_i = \mathbf{x}_i - \mathbf{x}_{i-1}$ for $i = 2, .., n$. Letting $\{\mathbf{E}_1, \mathbf{E}_2\}$ be a fixed orthonormal basis in the plane, we define ϕ_i as the angle that $\mathbf{E}_1$ has to rotate through to equal $\mathbf{e}_i$. The angles $\{\phi_i\}$ coordinitize the constrained chain. An alternate set of coordinates is given by $\{\theta_i\}$, where $\theta_1 = \phi_1$ and $\theta_i = \phi_i - \phi_{i-1}$ for $i = 2, ..., n$. These vectors and coordinates are illustrated in figure 1 below. The equations of motion for the chain consist of the functions $\ddot{\theta}_i = \ddot{\theta}_i(\dot{\theta}_i, \theta_i)$, and follow in our derivation

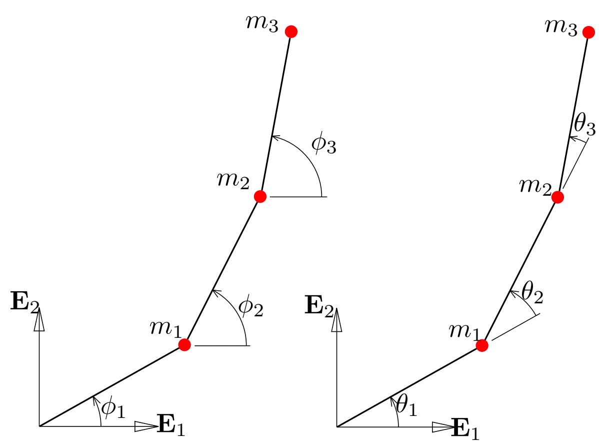

Figure 1: System coordinates.

from the balance law $f = ma$. In detail, we define the $2n$x1 vector $\mathbf{x}$ by $\mathbf{x} = [\mathbf{x}_1; \mathbf{x}_2; ...; \mathbf{x}_n]$, where semicolons are used in their Matlab sense. The vector $\mathbf{x}$ locates a 'super-particle', which we imagine tracing out a trajectory in $\mathbb{R}^{2n}$. Because of the constraints, $\mathbf{x}$ is restricted by an unknown 2nx1 constraint force $\mathbf{f}_c$ to a point p in an n-dimensional manifold M embedded in $\mathbb{R}^{2n}$. Additional forces denoted by $\mathbf{f}$ also act on the super-particle. The dynamics of the superparticle (and hence of the chain) are given by a balance of linear momentum

$$\mathbf{f} + \mathbf{f}_c = \mathbf{M}\ddot{\mathbf{x}} \tag{1}$$

where we define $\mathbf{M}$ =diag$(m_1, m_1, ..., m_n, m_n)$. The second derivatives of θ_i are included in the components of (1) in T_pM, which we note is spanned by the columns of $\mathbf{A} = \frac{\partial \mathbf{x}}{\partial \theta}$. Using normality to prescribe the constraint forces (i.e. choosing $\mathbf{f}_c$ so that $\mathbf{A}^T\mathbf{f}_c = \mathbf{0}$), (1) becomes

$$\mathbf{A}^T\mathbf{f} = \mathbf{A}^T\mathbf{M}\ddot{\mathbf{x}} \tag{2}$$

Now it only remains to prescribe the forces $\mathbf{f}$ which act on the chain. We have gravity (in the $\mathbf{E}_1$ direction) act on every mass, and we also impose moments at every joint, proportional to θ_i and $\dot{\theta}_i$ (corresponding to spring and damping forces respectively). Finally, we apply the follower force $-\alpha\mathbf{e}_n$ to m_n, where α is a tunable parameter.

2 Equations of Motion

Letting $s_m^n = \sum_{k=m}^n$, an expression for $\mathbf{x}$ is given by

$$\mathbf{x} = l \begin{bmatrix} s_1^1 \cos\phi_k \\ s_1^1 \sin\phi_k \\ s_1^2 \cos\phi_k \\ s_1^2 \sin\phi_k \\ s_1^3 \cos\phi_k \\ s_1^3 \sin\phi_k \\ \vdots \\ s_1^n \cos\phi_k \\ s_1^n \sin\phi_k \end{bmatrix} = l \begin{bmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ 1 & 0 & 1 & & & & & & \\ 0 & 1 & 0 & 1 & & & & & \\ 1 & 0 & 1 & 0 & 1 & & & & \\ 0 & 1 & 0 & 1 & 0 & 1 & & & \\ \vdots & & & & & & \ddots & & \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi_1 \\ \sin\phi_1 \\ \cos\phi_2 \\ \sin\phi_2 \\ \cos\phi_3 \\ \sin\phi_3 \\ \vdots \\ \cos\phi_n \\ \sin\phi_n \end{bmatrix} \tag{3}$$

Here and throughout these notes, expressions are given in more than one form when analytical brevity leads to expressions that are not easy to code. The matrix of ones and zeros above can be created in Matlab with the commands `P=[ones(1,n);zeros(1,n)]; P=tril(repmat([P(:)';1-P(:)'],n,1));`. Differentiating twice with respect to time, we find that $\ddot{x} = l\mathbf{A}\ddot{\theta} - l\mathbf{B}\dot{\phi}^2$, where $\ddot{\theta} = [\ddot{\theta}_1\ddot{\theta}_2\dots\ddot{\theta}_n]^T$, where $\dot{\phi}^2 = [\dot{\phi}_1^{\,2}\dot{\phi}_2^{\,2}...\dot{\phi}_n^{\,2}]^T$, where $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} -s_1^1 \sin\phi_k & 0 & 0 & \dots & 0 \\ s_1^1 \cos\phi_k & 0 & 0 & \dots & 0 \\ -s_1^2 \sin\phi_k & -s_2^2 \sin\phi_k & 0 & \dots & 0 \\ s_1^2 \cos\phi_k & s_2^2 \cos\phi_k & 0 & \dots & 0 \\ -s_1^3 \sin\phi_k & -s_2^3 \sin\phi_k & -s_3^3 \sin\phi_k & & 0 \\ s_1^3 \cos\phi_k & s_2^3 \cos\phi_k & s_3^3 \cos\phi_k & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s_1^n \sin\phi_k & -s_2^n \sin\phi_k & -s_3^n \sin\phi_k & \dots & -s_n^n \sin\phi_k \\ s_1^n \cos\phi_k & s_2^n \cos\phi_k & s_3^n \cos\phi_k & \dots & s_n^n \cos\phi_k \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ 1 & 0 & 1 & & & & & & \\ 0 & 1 & 0 & 1 & & & & & \\ 1 & 0 & 1 & 0 & 1 & & & & \\ 0 & 1 & 0 & 1 & 0 & 1 & & & \\ \vdots & & & & & & \ddots & & \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & \\ 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\phi_1 & 0 & 0 & \dots & 0 \\ \cos\phi_1 & 0 & 0 & \dots & 0 \\ -\sin\phi_2 & -\sin\phi_2 & 0 & \dots & 0 \\ \cos\phi_2 & \cos\phi_2 & 0 & \dots & 0 \\ -\sin\phi_3 & -\sin\phi_3 & -\sin\phi_3 & & 0 \\ \cos\phi_3 & \cos\phi_3 & \cos\phi_3 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sin\phi_n & -\sin\phi_n & -\sin\phi_n & \dots & -\sin\phi_n \\ \cos\phi_n & \cos\phi_n & \cos\phi_n & \dots & \cos\phi_n \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\phi_1 & 0 & 0 & \dots & 0 \\ \cos\phi_1 & 0 & 0 & \dots & 0 \\ -\sin\phi_1 & -\sin\phi_2 & 0 & \dots & 0 \\ \cos\phi_1 & \cos\phi_2 & 0 & \dots & 0 \\ -\sin\phi_1 & -\sin\phi_2 & -\sin\phi_3 & & 0 \\ \cos\phi_1 & \cos\phi_2 & \cos\phi_3 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sin\phi_1 & -\sin\phi_2 & -\sin\phi_3 & \dots & -\sin\phi_n \\ \cos\phi_1 & \cos\phi_2 & \cos\phi_3 & \dots & \cos\phi_n \end{bmatrix} \begin{bmatrix} 1 & & & & & & & & \\ 1 & 1 & & & & & & & \\ 1 & 1 & 1 & & & & & & \\ 1 & 1 & 1 & 1 & & & & & \\ 1 & 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & & & \\ \vdots & & & & & & \ddots & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

(note again that $\mathbf{A} = \frac{\partial \mathbf{x}}{\partial \theta}$ and so the columns of $\mathbf{A}$ span T_pM), and where $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} \cos\phi_1 & 0 & 0 & \dots & 0 \\ \sin\phi_1 & 0 & 0 & \dots & 0 \\ \cos\phi_1 & \cos\phi_2 & 0 & \dots & 0 \\ \sin\phi_1 & \sin\phi_2 & 0 & \dots & 0 \\ \cos\phi_1 & \cos\phi_2 & \cos\phi_3 & & 0 \\ \sin\phi_1 & \sin\phi_2 & \sin\phi_3 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cos\phi_1 & \cos\phi_2 & \cos\phi_3 & \dots & \cos\phi_n \\ \sin\phi_1 & \sin\phi_2 & \sin\phi_3 & \dots & \sin\phi_n \end{bmatrix} \tag{5}$$

We note that $\mathbf{x}$ is given by l times the sum of the columns of $\mathbf{B}$. The system dyanmics $\ddot{\theta}$ can now be found from

$$\mathbf{A}^T\mathbf{M}\mathbf{A}\ddot{\theta} = \mathbf{A}^T\mathbf{M}\mathbf{B}\dot{\phi}^2 + \frac{1}{l}\mathbf{A}^T\mathbf{f} \tag{6}$$

2.1 Kinetic Energy

The system kinetic energy is given by

$$KE = \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} = \dot{\theta}^T(\frac{l^2}{2}\mathbf{A}^T\mathbf{M}\mathbf{A})\dot{\theta} \tag{7}$$

2.2 Potential Energy

The system potential energy due to a gravitational body force acting on every mass in the $\mathbf{E}_1$ direction is given by

$$PE = gl(m_1 + 2m_2 + \dots + nm_n) - gl \begin{bmatrix} 0 & m_1 & 0 & m_2 & \dots & 0 & m_n \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{8}$$

where the first term causes PE to be zero when the chain is hanging straight down.

3 Forces

The force $\mathbf{f}$ acting on the superparticle is given by $\mathbf{f} = [\mathbf{f}_1; \mathbf{f}_2; ...; \mathbf{f}_n]$, where as with $\mathbf{x}$ we use semicolons in their Matlab sense, and where $\mathbf{f}_i$ is the force applied to mass m_i. Forces are applied to the masses due to gravitation, damping, and stiffness, and a follower force is applied to the very last mass.

3.1 Gravitation

The contribution to $\mathbf{f}$ due to gravity is given by $\mathbf{f}_g$ where

$$\mathbf{f}_g = g\mathbf{M} \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 \end{bmatrix}^T \tag{9}$$

3.2 Damping and Stiffness

The contributions to $\mathbf{f}$ due to damping and stiffness are given by $\mathbf{f}_d$ and $\mathbf{f}_s$ respectively where

$$\begin{aligned} \mathbf{f}_d &= \frac{1}{l}\mathbf{C}\mathbf{b}\dot{\theta} \\ \mathbf{f}_s &= \frac{1}{l}\mathbf{C}\mathbf{k}(\theta - \theta_{ref}) \end{aligned} \tag{10}$$

where $\mathbf{b}$ =diag$(b_1, \ldots, b_n)$ and $\mathbf{k}$ =diag$(k_1, \ldots, k_n)$ contain the damping and spring coifficients respectively for each of the n joints, and where $\mathbf{C}$ is given by

$$\mathbf{C} = \begin{bmatrix} \sin\phi_1 & -\sin\phi_1 - \sin\phi_2 & \sin\phi_2 & 0 & \ldots & 0 \\ -\cos\phi_1 & \cos\phi_1 + \cos\phi_2 & -\cos\phi_2 & 0 & \ldots & 0 \\ 0 & \sin\phi_2 & -\sin\phi_2 - \sin\phi_3 & \sin\phi_3 & & 0 \\ 0 & -\cos\phi_2 & \cos\phi_2 + \cos\phi_3 & -\cos\phi_3 & & 0 \\ 0 & 0 & \sin\phi_3 & -\sin\phi_3 - \sin\phi_4 & & 0 \\ 0 & 0 & -\cos\phi_3 & \cos\phi_3 + \cos\phi_4 & & 0 \\ 0 & 0 & 0 & \sin\phi_4 & & 0 \\ 0 & 0 & 0 & -\cos\phi_4 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & & \sin\phi_{n-1} \\ 0 & 0 & 0 & 0 & & -\cos\phi_{n-1} \\ 0 & 0 & 0 & 0 & & -\sin\phi_{n-1} - \sin\phi_n \\ 0 & 0 & 0 & 0 & & \cos\phi_{n-1} + \cos\phi_n \\ 0 & 0 & 0 & 0 & & \sin\phi_n \\ 0 & 0 & 0 & 0 & & -\cos\phi_n \end{bmatrix} \tag{11}$$

3.3 Follower Force

A follower force $\mathbf{f}_f = -\alpha\mathbf{e}_n$ is applied to the last mass m_n. Our interest is in the system stability as α is varied.

4 Linearization

When the angles θ_i and their time derivatives are small, certain terms are small enough to be neglected from the dynamics. In particular, the $\mathbf{A}^T\mathbf{MB}\dot{\phi}^2$ term from (6) drops out, and $\mathbf{A}^T\mathbf{MA}$ reduces to $\tilde{\mathbf{A}}^T\tilde{\mathbf{M}}\tilde{\mathbf{A}}$ where $\tilde{\mathbf{M}}$ =diag$(m_1, \ldots, m_n)$, and where

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 3 & 2 & 1 & & \\ \vdots & \vdots & & \ddots & \\ n & n-1 & n-2 & \ldots & 1 \end{bmatrix} \tag{12}$$

The term $\mathbf{A}^T\mathbf{f}_g$ due to gravity reduces to $-g\mathbf{G}(m_i)\theta$ where $\mathbf{G}(m_i)$ is given by

$$\mathbf{G}(m_i) = \left(\begin{bmatrix} m_1 & & \\ & & \\ & & \end{bmatrix} + \begin{bmatrix} m_2 & m_2 & \\ & m_2 & \\ & & \end{bmatrix} + \cdots + \begin{bmatrix} m_n & m_n & \cdots & m_n \\ & m_n & & \\ & & \ddots & \\ & & & m_n \end{bmatrix} \right) \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & \cdots & \cdots & 1 \end{bmatrix} \tag{13}$$

Also, $\frac{1}{l}\mathbf{A}^T\mathbf{Cb}\dot{\theta}$ reduces to $\frac{1}{l}\tilde{\mathbf{A}}^T\tilde{\mathbf{C}}\mathbf{b}\dot{\theta}$ and $\frac{1}{l}\mathbf{A}^T\mathbf{Ck}\theta$ reduces to $\frac{1}{l}\tilde{\mathbf{A}}^T\tilde{\mathbf{C}}\mathbf{k}\theta$ where $\tilde{\mathbf{C}}$ is the nxn matrix given by

$$\tilde{\mathbf{C}} = \begin{bmatrix} -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \\ & & & & & & -1 \end{bmatrix} \tag{14}$$

The follower force term $\frac{1}{l}\mathbf{A}^T\mathbf{f}_f$ reduces to $\frac{\alpha}{l}\mathbf{F}\theta$, where

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1-n \\ & 1 & \cdots & 1 & 2-n \\ & & \ddots & \vdots & \vdots \\ & & & 1 & -1 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \tag{15}$$

4.1 Linearized equations

Pulling the previous formulae together, we obtain the following linearized equations of motion

$$\tilde{\mathbf{A}}^T\tilde{\mathbf{M}}\tilde{\mathbf{A}}\ddot{\theta} = \left(-\frac{g}{l}\mathbf{G} + \frac{1}{l}\tilde{\mathbf{A}}^T\tilde{\mathbf{C}}\mathbf{k} + \frac{\alpha}{l}\mathbf{F}\right)\theta + \frac{1}{l}\tilde{\mathbf{A}}^T\tilde{\mathbf{C}}\mathbf{b}\dot{\theta} \tag{16}$$

when all masses have value m, all damping coefficients equal b, and all siffness coefficients equal k, these equations reduce further. In particular, $\mathbf{G}$ is given by

$$m \begin{bmatrix} n & n-1 & n-2 & \cdots & 1 \\ & n-1 & n-2 & \cdots & 1 \\ & & n-2 & \cdots & 1 \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \tag{17}$$

4.2 Linearized Equations with $n = 5$

A sense of the overal structure of the linearized equations can be obtained by examining the case $n = 5$:

$$\begin{aligned} & m \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 3 & 2 & 1 & & \\ 4 & 3 & 2 & 1 & \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \\ \ddot{\theta}_5 \end{bmatrix} \\ &= \frac{b}{l} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \\ & & & & -1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \end{bmatrix} \\ &+ \frac{k}{l} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 2 & 3 \\ & & & 1 & 2 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \\ & & & & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} \\ &- \left(\frac{gm}{l} \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ & 4 & 3 & 2 & 1 \\ & & 3 & 2 & 1 \\ & & & 2 & 1 \\ & & & & 1 \end{bmatrix} + \frac{\alpha}{l} \begin{bmatrix} 1 & 1 & 1 & 1 & -4 \\ & 1 & 1 & 1 & -3 \\ & & 1 & 1 & -2 \\ & & & 1 & -1 \\ & & & & 0 \end{bmatrix} \right) \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} \end{aligned} \tag{18}$$

4.3 Matlab Code for Generating the System Matrix

```
function D=Maciej(n,m,l,b,k,g,a)
%Maciej.m outputs the linearized dynamics D of a chain of n point masses
```

```
%where:
%       m is the value of each mass
%       l is the distance between adjacent masses
%       b is a damping coefficient (damping occurs at each joint)
%       k is a spring coefficient (each joing is sprung as well)
%       g is the value of gravity
%       a is a follower force parameter

Q=tril(ones(n));
P=triu(ones(n));
P(:,n)=[1-n:0]';
C=diag(2*ones(1,n-1),1)-diag(ones(1,n))-diag(ones(1,n-2),2);
A=tril([1:n]'*ones(1,n)-ones(n,1)*[0:n-1]);
G=tril(fliplr(ones(n,1)*[1:n])');
D=[zeros(n) eye(n);
    (A'*A)\(k/l/m*A'*C-(g/l*G'-a/l/m*P)*Q) (A'*A)\A'*C*b/l/m];
```

5 The Case $n = 2$

When $n = 2$ the nonlinear equations of motion are given by

$$\mathbf{A}^T\mathbf{M}\mathbf{A}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} = \mathbf{A}^T\mathbf{M}\mathbf{B}\begin{bmatrix}\dot{\theta}_1^2\\ (\dot{\theta}_1+\dot{\theta}_2)^2\end{bmatrix} + \frac{1}{l}\mathbf{A}^T(\mathbf{C}\mathbf{b}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} + \mathbf{C}\mathbf{k}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + \mathbf{f}_g + \mathbf{f}_f) \tag{19}$$

where

$$\mathbf{A} = \begin{bmatrix} -\sin\theta_1 & 0\\ \cos\theta_1 & 0\\ -\sin\theta_1 - \sin(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \cos\theta_1+\cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{bmatrix} \mathbf{M} = \begin{bmatrix} m_1 & & & \\ & m_1 & & \\ & & m_2 & \\ & & & m_2\end{bmatrix} \tag{20}$$

$$\mathbf{B} = \begin{bmatrix}\sin\theta_1 & 0\\ \cos\theta_1 & 0\\ \sin\theta_1 & \sin(\theta_1+\theta_2)\\ \cos\theta_1 & \cos(\theta_1+\theta_2)\end{bmatrix} \mathbf{C} = \begin{bmatrix}\sin\theta_1 & -\sin\theta_1-\sin(\theta_1+\theta_2)\\ -\cos\theta_1 & \cos\theta_1+\cos(\theta_1+\theta_2)\\ 0 & \sin(\theta_1+\theta_2)\\ 0 & -\cos(\theta_1+\theta_2)\end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix}b_1 & \\ & b_2\end{bmatrix} \mathbf{k} = \begin{bmatrix}k_1 & \\ & k_2\end{bmatrix} \mathbf{f}_g = g\begin{bmatrix}m_1\\ 0\\ m_2\\ 0\end{bmatrix} \mathbf{f}_f = -\alpha\begin{bmatrix}0\\ 0\\ \cos(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2)\end{bmatrix}$$

We note that in this case

$$\mathbf{A}^T\mathbf{M}\mathbf{A} = \begin{bmatrix} m_1+2m_2(1+\cos\theta_2) & m_2(1+\cos\theta_2)\\ m_2(1+\cos\theta_2) & m_2\end{bmatrix} \tag{21}$$

$$\mathbf{A}^T\mathbf{M}\mathbf{B} = m_2\sin\theta_2\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$$

which is in agreement with a derivation of the dynamics using Lagrange's equations. These nonlinear algebraic simplifications probably hold in general, but we haven't yet worked them out.

5.1 Linearization

A linearization of the above equations assuming uniform mass, stiffness, and damping values gives

$$ml\begin{bmatrix}5 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} = \begin{bmatrix}2\alpha-k-3mg & \alpha-mg\\ -mg & -k-mg\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + \begin{bmatrix}-b & 0\\ 0 & -b\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} \tag{22}$$

6 Simulation Code

```
function planarchain
%planarchain.m solves for the dynamics of a chain of point masses confined
%to a plane.  parameters that can be varied include:
%
%       n is the number of point masses in the chain
%       m is the value of each mass
%       l is the distance between adjacent masses
%       b is a damping coefficient (damping occurs at each joint)
%       k is a spring coefficient (each joint is sprung as well)
%       g is the value of gravity
%       a is a follower force parameter

%establish parameters:
n=5;
m=1e-2;
l=0.1;
b=0.005;
k=0;%0.001;
g=9.8;
a=1.9*m*g;  %stability threshhold for these values seems to be about 1.9

%establish desired time interval and initial configuration:
time=linspace(0,1,500);
thin=0.0001*ones(1,n);     %theta values
thdin=zeros(1,n);    %theta derivatives

%solve the ODE.
%@chain solves the nonlinear and @linchain solves the linear equations.
if 0    %Solve the nonlinear equations
    opt=odeset('RelTol',1e-4);
    [t,state]=ode45(@chain,time,[thin thdin]',opt,n,m,l,b,k,g,a);
else    %Solve the linear equations
    opt=odeset('RelTol',1e-4);
    SYS=Maciej(n,m,l,b,k,g,a);
    [t,state]=ode45(@linchain,time,[thin thdin]',opt,SYS);
end
th=state(:,1:n);
thd=state(:,n+1:end);

%animate!
fig1=figure;
set(fig1,'color',[1 1 1],'backingstore','off','Doublebuffer','on');
for i=1:length(t)
    %construct a 2xN array of position vectors
    x=zeros(2,n);
    x(:,1)=l*[cos(th(i,1));sin(th(i,1))];
    for j=2:n
        x(:,j)=x(:,j-1)+l*[cos(sum(th(i,1:j)));sin(sum(th(i,1:j)))];
    end
    plot([0 x(1,:)],[0 x(2,:)],'b',x(1,:),x(2,:),'ro',...
         0.5*[-l l],[0 0],'k',[0 0],0.5*[-l l],'k');
    axis equal
```

```
    xlim([-l*n,l*n])
    ylim([-l*n,l*n])
    drawnow
end

%Energy Plots:
PE=zeros(length(t),1);
KE=zeros(length(t),1);
for i=1:length(t)
    thc=th(i,:)';    thdc=thd(i,:)';
    ph=zeros(n,1);  phd=zeros(n,1);
    ph(1)=thc(1);   phd(1)=thdc(1);
    for j=2:n
        ph(j)=ph(j-1)+thc(j);    phd(j)=phd(j-1)+thdc(j);
    end
    sines=sin(ph);  cosines=cos(ph);
    B1=tril(ones(n,1)*cosines');
    B2=tril(ones(n,1)*sines');
    B=[B1(:) B2(:)]';
    B=reshape(B(:),2*n,n);
    A=[-B2(:) B1(:)]';
    A=reshape(A(:),2*n,n)*tril(ones(n));
    Q=[zeros(1,n);ones(1,n)];
    PE(i)=m*g*l*(0.5*n*(n+1)-Q(:)'*A(:,1));
    KE(i)=0.5*m*l^2*thdc'*(A'*A)*thdc;
end
fig2=figure;
set(fig2,'color',[1 1 1])
plot(time,KE,'r:',time,PE,'b:',time,KE+PE,'k')
legend('Kinetic Energy','Potential Energy','Total Energy')
xlabel('time')
ylabel('energy')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function dstate=chain(t,state,n,m,l,b,k,g,a)
%nonlinear equations of motion:
th=state(1:n);  thd=state(n+1:2*n);
ph=zeros(n,1);  phd=zeros(n,1);
ph(1)=th(1);    phd(1)=thd(1);
for i=2:n
    ph(i)=ph(i-1)+th(i);    phd(i)=phd(i-1)+thd(i);
end
sines=sin(ph);  cosines=cos(ph);
B1=tril(ones(n,1)*cosines');
B2=tril(ones(n,1)*sines');
B=[B1(:) B2(:)]';
B=reshape(B(:),2*n,n);
A=[-B2(:) B1(:)]';
A=reshape(A(:),2*n,n)*tril(ones(n));
%Now for the forces:
C=zeros(2*n,n);
C(1:2,1)=[sines(1);-cosines(1)];
C(1:4,2)=[-sines(1)-sines(2);cosines(1)+cosines(2);sines(2);-cosines(2)];
for i=3:n
```

```
    C(2*i-5:2*i,i)=[sines(i-1);-cosines(i-1);-sines(i-1)-sines(i);
                    cosines(i-1)+cosines(i);sines(i); -cosines(i)];
end
fg=[ones(1,n);zeros(1,n)];
fg=m*g*fg(:);
ff=zeros(2*n,1);
ff(2*n-1:2*n,1)=-a*[cosines(n);sines(n)];
%Dynamics:
dstate=[thd;
        (A'*A)\(A'*B*phd.^2+(1/m/l)*A'*(ff+fg+C*((b/l)*thd +(k/l)*th)))];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function dstate=linchain(t,state,SYS)
%linearized equations of motion:
dstate=SYS*state;
```