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Rotation

My interest is in the rotation of physical objects in three dimensional space. Lots of interesting questions are raised in this attempt to explain everything from the ground up; in particular, the generalized cross and box products remind me of material I've seen on forms (which I don't yet understand).

1 Euclidean Vector Spaces

We use the term *Euclidean vector space* to denote an inner product space over $\mathbb{R}$.

1.1 Orientation

A basis for an n-dimensional vector space V is a list (lists have order) of n vectors. Two bases $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of a Euclidean vectors space are said to have the same *orientation* if the determinant of $[\mathbf{a}_i \bullet \mathbf{b}_j]$ is positive. Orientation is an equivalence relation that partitions the bases of the Euclidean vector space into two equivalence classes. The space is said to be *oriented* if these classes are distinguished; we do this by calling bases in one of them *right-handed* and bases in the other *left-handed*, (the choice of labels is unimportant so long as a choice is made one way or the other). From now on we assume that all Euclidean vector spaces have been oriented.

1.2 The Cross Product

Let $(\mathbf{E}_i)$ be any right-handed orthonormal basis for an n-dimensional Euclidean vector space V. We define the *cross product* as the multilinear mapping from V^{n-1} to V which assigns the ordered list $(\mathbf{a}_2, \cdots, \mathbf{a}_n)$ of $n-1$ vectors to the vector

$$\operatorname{cross}(\mathbf{a}_2, \cdots, \mathbf{a}_n) := \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \cdots & \mathbf{E}_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \tag{1}$$

where $\mathbf{a}_i = a_{ij}\mathbf{E}_j$. In a 3-dimensional Euclidean vector space, we denote $\operatorname{cross}(\mathbf{a}, \mathbf{b})$ by $\mathbf{a} \times \mathbf{b}$.

1.3 The Box Product

The *Box Product* $[\mathbf{a}_1, \cdots, \mathbf{a}_n]$ of a list $(\mathbf{a}_i)$ of n vectors in an n-dimensional Euclidean vector space is given by

$$[\mathbf{a}_1, \cdots, \mathbf{a}_n] = \mathbf{a}_1 \bullet \operatorname{cross}(\mathbf{a}_2, \cdots, \mathbf{a}_n) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \tag{2}$$

and can be thought of as the (signed) volume of the parallelipiped with edges $\mathbf{a}_i$. We note that $[\mathbf{a}_1, \cdots, \mathbf{a}_n] = [\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_n}]$ where $(i_1, \cdots, i_n)$ is any even permutation of $(1, \cdots, n)$, (a permutation is even if it can be obtained by an even number of pair-wise interchanges).

1.4 Determinant of an Operator

We define the determinant of an operator $\mathbf{T}$ on an n-dimensional Euclidean vector space to be the real number $\det(\mathbf{T})$ for which

$$\det(\mathbf{T})[\mathbf{a}_1, \cdots, \mathbf{a}_n] = [\mathbf{T}\mathbf{a}_1, \cdots, \mathbf{T}\mathbf{a}_n], \tag{3}$$

for every set $\{\mathbf{a}_i\}$ of n vectors in the vector space.

2 Rotations and Rotation Operators

A *rotation* in a three dimensional Euclidean vector space E^3 is the pairing of a unit vector $\mathbf{u}_1$ with an angle θ. We think of this pairing as an instruction; "rotate through an angle θ about the $\mathbf{u}_1$ direction". Without further structure though, this instruction is ambiguous. If $(\mathbf{u}_i)$ is an orthonormal basis for E^3, it isn't clear whether $\mathbf{u}_2$ is mapped to $\mathbf{u}_2 \cos\theta + \mathbf{u}_3 \sin\theta$ or to $\mathbf{u}_2 \cos\theta - \mathbf{u}_3 \sin\theta$. We resolve this question by adopting the convention that if $(\mathbf{u}_i)$ is a *right-handed* orthonormal basis, then $\mathbf{u}_2$ is mapped to $\mathbf{u}_2 \cos\theta + \mathbf{u}_3 \sin\theta$.

Every rotation $(\mathbf{u}, \theta)$ corresponds to a linear operator on E^3 that we call a *rotation operator*, and that we denote by $\mathbf{Q}(\mathbf{u}, \theta)$. A convenient formula for the mapping from a rotation $(\mathbf{u}, \theta)$ to its corresponding rotation operator $\mathbf{Q}(\mathbf{u}, \theta)$ is given by

$$\mathbf{Q}(\mathbf{u}, \theta) = \mathbf{u} \otimes \mathbf{u} + (\mathbf{I} - \mathbf{u} \otimes \mathbf{u}) \cos\theta + \mathbf{S}_{\mathbf{u}} \sin\theta, \tag{4}$$

where $\mathbf{S}_{\mathbf{u}}$ is the unique operator on E^3 which satisfies $\mathbf{S}_{\mathbf{u}}\mathbf{x} = \mathbf{u} \times \mathbf{x}$ for every $\mathbf{x} \in E^3$. The matrix of $\mathbf{Q}(\mathbf{u}, \theta)$ with respect to a right-handed orthonormal basis $(\mathbf{E}_i)$ is given by

$$\begin{aligned} M(\mathbf{Q}(\mathbf{u}, \theta), (\mathbf{E}_i)) &= \begin{bmatrix} u_1u_1(1-\cos\theta) + \cos\theta & u_1u_2(1-\cos\theta) - u_3\sin\theta & u_1u_3(1-\cos\theta) + u_2\sin\theta \\ u_2u_1(1-\cos\theta) + u_3\sin\theta & u_2u_2(1-\cos\theta) + \cos\theta & u_2u_3(1-\cos\theta) - u_1\sin\theta \\ u_3u_1(1-\cos\theta) - u_2\sin\theta & u_3u_2(1-\cos\theta) + u_1\sin\theta & u_3u_3(1-\cos\theta) + \cos\theta \end{bmatrix} \\ &= (1-\cos\theta) \begin{bmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \end{aligned} \tag{5}$$

If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors for which $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a right-handed orthonormal basis for E^3, then $\mathbf{Q}(\mathbf{u}, \theta)$ can be written as

$$\mathbf{Q}(\mathbf{u}, \theta) = (\mathbf{v} \otimes \mathbf{v} + \mathbf{w} \otimes \mathbf{w}) \cos\theta + (\mathbf{w} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{w}) \sin\theta + \mathbf{u} \otimes \mathbf{u}, \tag{6}$$

`RotationtoQ.m`

```
function Q=RotationtoQ(u,th)
%RtoM.m accepts a unit vector u and an angle th, and outputs the
%unique corresponding rotation matrix Q.

u=[u(1);u(2);u(3)]/sqrt(u(1)^2+u(2)^2+u(3)^2);
Q=(1-cos(th))*u*u'+cos(th)*eye(3)...
  +sin(th)*[0 -u(3) u(2);u(3) 0 -u(1);-u(2) u(1) 0];
```

3 From Rotation Operators to Rotations

The mapping from rotations to rotation operators is onto but not one-to-one. For instance,

$$\begin{aligned} \mathbf{Q}(\mathbf{u}, \theta) &= \mathbf{Q}(\mathbf{u}, \theta + 2\pi n) \\ &= \mathbf{Q}(-\mathbf{u}, -\theta + 2\pi n) \end{aligned} \tag{7}$$

for any integer n. To define a unique rotation $(\mathbf{u}, \theta)$ for any given rotation operator $\mathbf{Q}$, we must restrict our attention to a subset of the possible rotations, (so that a bijection exists between this subset and the set of all rotation operators). In these notes we do this by restricting θ to the closed interval $[0, \pi]$, with additional restrictions on $\mathbf{u}$ when $\theta = 0$ or $\theta = \pi$. This θ can be computed from the trace of the matrix in (5) using an arc-cosine function with a range of $[0, \pi]$.

- In the case $\theta \in (0, \pi)$, a unique $\mathbf{u}$ follows from the skew-symmetric part of (5).
- In the case $\theta = 0$, $\mathbf{Q}$ is the identity, and any unit vector for $\mathbf{u}$ will do. We choose $\mathbf{u} = \mathbf{E}_1$, where $(\mathbf{E}_i)$ is the basis from the previous section.

- In the case $\theta = \pi$, there are only two possible $\mathbf{u}$'s to choose from, each the negative of the other. We always choose the $\mathbf{u}$ for which $\mathbf{u} \cdot \mathbf{E}_3 \geq 0$. If $\mathbf{u} \cdot \mathbf{E}_3 = 0$, then we choose the $\mathbf{u}$ for which $\mathbf{u} \cdot \mathbf{E}_2 \geq 0$. Finally, if $\mathbf{u} \cdot \mathbf{E}_2$ and $\mathbf{u} \cdot \mathbf{E}_2$ are zero, we choose $\mathbf{u} = \mathbf{E}_1$.

QtoRotation.m

```
function [u,th]=QtoRotation(Q)
%MtoR.m accepts a proper-orthogonal Q, and returns a vector u and angle th.

tol=1e-10;
if sum(sum(abs(Q-eye(3))))<tol %Q is the identity
    th=0;
    u=[1;0;0];
    disp(['Warning: Q is within ' num2str(tol) ' of the identity.'])
    return
end
c=0.5*(Q(1,1)+Q(2,2)+Q(3,3)-1);        %cos(theta)
S=0.5*(Q-Q');
if sum(abs([S(3,2);S(1,3);S(2,1)]))<tol  %Q is a 180 deg rotation
    th=pi;
    u=sqrt(0.5*([Q(1,1);Q(2,2);Q(3,3)]+1));
    nzero=find(Q(:,1)~=0);
    i1=nzero(1);
    i2=mod(i1+1,3);
    i3=mod(i1+2,3);
    u(i2)=u(i2)*sign(Q(i2,i1))*sign(u(i1));
    u(i3)=u(i3)*sign(Q(i3,i1))*sign(u(i1));
    if u(3)<0 | (u(3)==0 & (u(2)<0 | (u(1)==0 & (u(1)<0)))) %uniqueness.
        u=-u;
    end
    disp(['Warning: Q is within ' num2str(tol) ' of a 180 deg rotation.'])
    return
end
th=acos(c);
u=[S(3,2);S(1,3);S(2,1)]/sin(c);
```

4 Rotation Operators and Proper-Orthogonal Operators

An operator on E^3 is called *proper* if its determinant equals 1, and *orthogonal* if it has an inverse that is equal to its transpose. We now show that the set of rotation operators equals the set of proper-orthogonal operators.

Rotation Operators $\subset$ Proper-Orthogonal Operators

We use (6) to show that the rotation operator $\mathbf{Q}(\mathbf{u}, \theta)$ is proper-orthogonal. Let $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ be a right-handed orthonormal basis for E^3, and note that

$$\mathbf{Q}(\mathbf{u}, \theta) = \tilde{\mathbf{u}} \otimes \mathbf{u} + \tilde{\mathbf{v}} \otimes \mathbf{v} + \tilde{\mathbf{w}} \otimes \mathbf{w}, \tag{8}$$

where

$$\tilde{\mathbf{u}} = \mathbf{Q}(\mathbf{u}, \theta)\mathbf{u} = \mathbf{u}, \qquad \tilde{\mathbf{v}} = \mathbf{Q}(\mathbf{u}, \theta)\mathbf{v} = \mathbf{v}\cos\theta + \mathbf{w}\sin\theta, \qquad \tilde{\mathbf{w}} = \mathbf{Q}(\mathbf{u}, \theta)\mathbf{w} = -\mathbf{v}\sin\theta + \mathbf{w}\cos\theta. \tag{9}$$

A computation reveals that $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, \tilde{\mathbf{w}})$ is right-handed and orthonormal, and then it follows from (8) that $\mathbf{Q}(\mathbf{u}, \theta)$ is proper-orthogonal.

Proper-Orthogonal Operators $\subset$ Rotation Operators

Our argument that every proper-orthogonal $\mathbf{Q}$ can be written as a rotation operator $\mathbf{Q}(\mathbf{u}, \theta)$ for some unit vector $\mathbf{u}$ and some angle θ comes from a handout by Professor Papadopoulos that he distributed to his ME185 class in 2001. Start with a proper-orthogonal $\mathbf{Q}$ and note that

$$\begin{aligned}
\mathbf{Q}^T\mathbf{Q} = \mathbf{I} &\Longrightarrow \mathbf{Q}^T\mathbf{Q} - \mathbf{Q}^T = \mathbf{I} - \mathbf{Q}^T,\\
&\Longrightarrow \mathbf{Q}^T(\mathbf{Q} - \mathbf{I}) = -(\mathbf{Q} - \mathbf{I})^T,\\
&\Longrightarrow \det\mathbf{Q}\det(\mathbf{Q} - \mathbf{I}) = \det(-(\mathbf{Q} - \mathbf{I})^T) = -\det(\mathbf{Q} - \mathbf{I}),\\
&\Longrightarrow \det(\mathbf{Q} - \mathbf{I}) = -\det(\mathbf{Q} - \mathbf{I}),\\
&\Longrightarrow \det(\mathbf{Q} - \mathbf{I}) = 0,\\
&\Longrightarrow 1 \in \operatorname{Spec}(\mathbf{Q}), \qquad (10)
\end{aligned}$$

where we've used the fact that $\det(\mathbf{A}^T) = \det\mathbf{A}$ and that $\det(-\mathbf{A}) = -\det\mathbf{A}$ for any operator $\mathbf{A}$ on E^3. Let $\mathbf{u}_1$ be a unit eigenvector of $\mathbf{Q}$ associated with its unit eigenvalue. Pick unit vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ so that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is a right-handed orthonormal basis for E^3. We know that $\mathbf{Q}$ can be written as $Q_{ij}\mathbf{u}_i \otimes \mathbf{u}_j$. We now solve for the Q_{ij}'s

- $\mathbf{Q}\mathbf{u}_1 = \mathbf{u}_1$, and so $Q_{11} = 1$, $Q_{21} = 0$, and $Q_{31} = 0$.
- $\mathbf{Q}\mathbf{u}_1 = \mathbf{u}_1 \Longrightarrow \mathbf{u}_1 = \mathbf{Q}^T\mathbf{u}_1$, (because $\mathbf{Q}^{-1} = \mathbf{Q}^T$), and so $Q_{12} = 0$ and $Q_{13} = 0$.
- With the above findings in mind, we expand $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and find that the $\mathbf{u}_2 \otimes \mathbf{u}_2$ and $\mathbf{u}_3 \otimes \mathbf{u}_3$ terms give

  $$\begin{aligned}
  &Q_{22}^2 + Q_{23}^2 = 1 \Longrightarrow \text{Some } \theta \text{ exists for which } \cos\theta = Q_{22} \text{ and } \sin\theta = Q_{23}.\\
  &Q_{32}^2 + Q_{33}^2 = 1 \Longrightarrow \text{Some } \phi \text{ exists for which } \sin\phi = Q_{32} \text{ and } \cos\phi = Q_{33}.
  \end{aligned}$$

- The $\mathbf{u}_2 \otimes \mathbf{u}_3$ and the $\mathbf{u}_3 \otimes \mathbf{u}_2$ terms both give us $Q_{22}Q_{32} + Q_{23}Q_{33} = 0$, which implies that $\cos\theta\sin\phi + \cos\phi\sin\theta = \sin(\phi + \theta) = 0$. This in turn implies that $\phi + \theta = n\pi$ for some integer n, and so we are able to put everything in terms of θ; $Q_{32} = \cos\phi = (-1)^{|n|}\sin\theta$ and $Q_{33} = \sin\phi = (-1)^{|n|+1}\cos\theta$. The resulting expression for $\mathbf{Q}$ is then given by

  $$\mathbf{Q} = \mathbf{u}_1 \otimes \mathbf{u}_1 + \cos\theta(\mathbf{u}_2 \otimes \mathbf{u}_2 + (-1)^{|n|+1}\mathbf{u}_3 \otimes \mathbf{u}_3) + \sin\theta(\mathbf{u}_2 \otimes \mathbf{u}_3 + (-1)^{|n|}\mathbf{u}_3 \otimes \mathbf{u}_2) \qquad (11)$$

- Note that $\det\mathbf{Q} = (-1)^{|n|+1}$, and so for $\mathbf{Q}$ to be proper, it must be that $|n|$ is odd. This causes (11) to become

  $$\mathbf{Q} = \mathbf{u}_1 \otimes \mathbf{u}_1 + \cos\theta(\mathbf{u}_2 \otimes \mathbf{u}_2 + \mathbf{u}_3 \otimes \mathbf{u}_3) + \sin\theta(\mathbf{u}_2 \otimes \mathbf{u}_3 - \mathbf{u}_3 \otimes \mathbf{u}_2) \qquad (12)$$

  which is the same form as (6) if we replace θ with $-\theta$.

We have shown that $\mathbf{Q}$ can be written in the same form as (6), and so $\mathbf{Q}$ is a rotation operator.

4.1 Eigenvalues and Eigenvectors

In Section 3 we showed how to find a rotation $(\mathbf{u}, \theta)$ corresponding to a proper-orthogonal (i.e., rotation) operator $\mathbf{Q}$. In this section we do the same thing, but with eigenvalues and eigenvectors. We start by moving to the vector space over $\mathbb{C}$ consisting of the ordered pairs $(\mathbf{u}, \mathbf{v})$ where $\mathbf{u}, \mathbf{v} \in E^3$, (we often use $\mathbf{u} + i\mathbf{v}$ to denote $(\mathbf{u}, \mathbf{v})$). Addition is defined componentwise and scalar multiplication is defined by $(a + ib)(\mathbf{u} + i\mathbf{v}) = a\mathbf{u} - b\mathbf{v} + i(a\mathbf{v} + b\mathbf{u})$. We call this vector space the *complexification* of E^3 and we denote it by F^3. A conjugate-symmetric inner product on F^3 is given by

$$\langle \mathbf{u} + i\mathbf{v}, \mathbf{w} + i\mathbf{z}\rangle = \mathbf{u} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{z} + i(\mathbf{v} \bullet \mathbf{w} - \mathbf{u} \bullet \mathbf{z}), \qquad (13)$$

where $\mathbf{u} \bullet \mathbf{v}$ is the inner product of $\mathbf{u}, \mathbf{v} \in E^3$.

We define the complexification of $\mathbf{Q} \in L(E^3)$ to be the operator $\mathbf{R} \in L(F^3)$ that maps $\mathbf{u} + i\mathbf{v}$ to $\mathbf{Q}\mathbf{u} + i\mathbf{Q}\mathbf{v}$. Because $\mathbf{Q} \in L(E^3)$ is orthogonal, $(\mathbf{Q}\mathbf{u}) \bullet (\mathbf{Q}\mathbf{v}) = \mathbf{u} \bullet \mathbf{v}$ for every $\mathbf{u}, \mathbf{v} \in E^3$. It follows from (13) that $\langle \mathbf{R}\mathbf{a}, \mathbf{R}\mathbf{a}\rangle = \langle \mathbf{a}, \mathbf{b}\rangle$ for all $\mathbf{a}, \mathbf{b} \in F^3$, that is, that $\mathbf{R} \in L(F^3)$ is orthogonal. An orthogonal operator is normal, and so from the complex spectral theorem, we know that F^3 has an orthonormal basis B of eigenvectors of $\mathbf{R}$. In fact, in terms of the basis $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ and the angle θ in (11), we can write

$$\begin{aligned} \mathbf{R}\mathbf{u}_1 &= \mathbf{u}_1 \\ \mathbf{R}(\mathbf{u}_2 - i\mathbf{u}_3) &= e^{i\theta}(\mathbf{u}_2 - i\mathbf{u}_3) \\ \mathbf{R}(\mathbf{u}_2 + i\mathbf{u}_3) &= e^{-i\theta}(\mathbf{u}_2 + i\mathbf{u}_3), \end{aligned} \tag{14}$$

and so θ and $\mathbf{u}_1$ can be recovered from a numerical routine that returns the (complex) eigenvalues λ_i and eigenvectors $\mathbf{a}_i$ of $\mathbf{R}$.

- If all three λ_i's equal 1, then $\mathbf{Q} = \mathbf{I}$.
- If λ_1 and λ_2 equal i, then $\theta = \pi$ and $\mathbf{u}_1$ is in the direction of $\mathbf{a}_3$, (a routine for choosing $\mathbf{u}_1$ uniquely can be used as in Section 3).
- If $\mathrm{Im}(\lambda_1) > 0$ and $\mathrm{Im}(\lambda_2) < 0$, then $\lambda_1 = e^{i\theta}$ for some $\theta \in [0, \pi]$, (this is the θ we're after). Of course $\mathbf{u}_1$ is in the direction of the real vector $\mathbf{a}_3$, however to choose between $+\mathbf{a}_3$ and $-\mathbf{a}_3$ we need to consider the orientation (in E^3) of the basis $(\mathrm{Re}(\mathbf{a}_1), \mathrm{Im}(\mathbf{a}_1), \mathbf{a}_3)$. If this basis is left-handed, then $\mathbf{u}_1 = \mathbf{a}_3$, and if this basis is right handed then $\mathbf{u}_1 = -\mathbf{a}_3$

`eigMtoR.m`

```
function [u,th]=eigMtoR(Q)
%MtoR.m accepts a proper-orthogonal Q, and returns a vector u and angle th,
%computed using eigenvalues and eigenvectors, assuming Q is neither the
%identity nor a 180 degree rotation.

[evec,eval]=eig(Q);
eval=diag(eval);
tol=1e-10;
i1=find(abs(eval-1)<tol);
u=real(evec(:,i1));
i2=mod(i1+1,3);
i3=mod(i1+2,3);
if imag(eval(i2))<0
    it=i2;  i2=i3;  i3=it;
end
th=acos(real(eval(i2))/abs(eval(i2)));
if det([u real(evec(:,i2)) imag(evec(:,i2))])>0
    u=-u;
end
```

5 Angular Velocity

If $\mathbf{Q}$ is a proper-orthogonal function of t, then $\mathbf{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T$ is skew-symmetric, and there exists a unique $\boldsymbol{\omega} \in E^3$ such that $\mathbf{\Omega}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}$ for every $\mathbf{x} \in E^3$. We call $\boldsymbol{\omega}$ the *angular velocity vector* associated with $\mathbf{Q}$. If the unit vector $\mathbf{u}$ and the angle θ are functions of t, then the $\boldsymbol{\omega}$ associated with the unique time dependant rotation operater $\mathbf{Q}(\mathbf{u}, \theta)$ is given by

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{u} + (1 - \cos\theta)\mathbf{u} \times \dot{\mathbf{u}} + \sin\theta\dot{\mathbf{u}}. \tag{15}$$

We note that $\boldsymbol{\omega}$ can be used to move through the rotations along a specified path $(\mathbf{u}(t), \theta(t))$. In detail, given any $\mathbf{u}$ and $\theta \neq 2\pi n$ for $n \in \mathbb{Z}$, we can choose $\boldsymbol{\omega}$ so as to achieve any $\dot{\mathbf{u}}$ and $\dot{\theta}$ we wish. When $\theta = 2\pi n$, recall that $\mathbf{Q}(\mathbf{u}, \theta) = \mathbf{I}$ and the choice of $\mathbf{u}$ is not unique. A nonzero $\boldsymbol{\omega}$ at such a point causes $\mathbf{u}$ to jump to the unit vector in the direction of $\boldsymbol{\omega}$.

Question: does a straight line path $(\mathbf{u}(t), \theta(t))$ between $(\mathbf{u}_1, \theta_1)$ and $(\mathbf{u}_2, \theta_2)$ correspond to the obvious minimum path between the two?

6 Rigid Bodies

Proper-Orthogonal operators play a central role in dynamics because they describe the orientation of rigid bodies in space. We establish this fact here.

A body is called *rigid* if its material particles always stay the same distance apart. If $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ locate material particles of a rigid body in one configuration and if $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ locate the same particles in a different configuration, then

$$\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{X}-\mathbf{Y}\|, \qquad \|\mathbf{x}-\mathbf{z}\| = \|\mathbf{X}-\mathbf{Z}\|, \qquad \|\mathbf{y}-\mathbf{z}\| = \|\mathbf{Y}-\mathbf{Z}\|. \tag{16}$$

From the polarization identity $2\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2$, we see that (16) gives

$$(\mathbf{X}-\mathbf{Y})\cdot(\mathbf{X}-\mathbf{Z}) = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{z}). \tag{17}$$

Let $\mathbf{P}$ be an interior point of a rigid body in one configuration (which we will call its reference configuration), and let $\mathbf{p}$ be the corresponding point in a different configuration (which we will call its current configuration). If $(\mathbf{E}_i)$ is some right-handed orthonormal basis for E^3, then for small enough l, $\mathbf{P}+l\mathbf{E}_i$ locate material points of the rigid body in its reference configuration. The corresponding locations in the current configuration are $\mathbf{P}+l\mathbf{e}_i$, for some list $(\mathbf{e}_i)$ of vectors in E^3. Using (17), we see that $\mathbf{e}_i\cdot\mathbf{e}_j = \mathbf{E}_i\cdot\mathbf{E}_j = \delta_{ij}$, and so $(\mathbf{e}_i)$ is an orthonormal basis for E^3, and it follows that $[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$ is either +1 or -1. Continuity in the motion of the body implies continuity in the function $[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]$. The function has value +1 when the $\mathbf{e}_i$'s coincide with the $\mathbf{E}_i$'s (i.e., when the body occupies its reference configuration), and so the function must equal +1 for all configurations. That is, the basis $(\mathbf{e}_i)$ is right-handed.

If $\mathbf{X}$ is the location occupied by an arbitrary material particle in the reference configuration, and if $\mathbf{x}$ is the location occupied by the same material particle in the current configuration, then from (17) we see that

$$(\mathbf{X}-\mathbf{P})\cdot\mathbf{E}_i = (\mathbf{x}-\mathbf{p})\cdot\mathbf{e}_i \Longrightarrow (\mathbf{x}-\mathbf{p}) = \mathbf{Q}(\mathbf{X}-\mathbf{P}), \tag{18}$$

where $\mathbf{Q} = \mathbf{e}_i\otimes\mathbf{E}_i$. It is straightforward to check that this $\mathbf{Q}$ is proper-orthogonal; note that $\mathbf{Q}^T = \mathbf{e}_i\otimes\mathbf{E}_i$, and $\mathbf{Q}\mathbf{Q}^T = (\mathbf{E}_i\otimes\mathbf{e}_i)(\mathbf{e}_j\otimes\mathbf{E}_j) = (\mathbf{E}_i\otimes\mathbf{E}_i)\delta_{ij} = \mathbf{I}$. Similarly, $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ as well, and so $\mathbf{Q}$ is orthogonal. The determinant of $\mathbf{Q}$ is the ratio of the scalar triple products $[\mathbf{Q}\mathbf{e}_1,\mathbf{Q}\mathbf{e}_2,\mathbf{Q}\mathbf{e}_3] = [\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3] = 1$ and $[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3] = 1$. This ratio is 1, and so $\mathbf{Q}$ is proper.

Thus, the point $\mathbf{x}$ that a particle of a rigid body occupies in its current configuration is related to the point $\mathbf{X}$ that the particle occupies in the reference configuration according to a transformation of the form

$$\mathbf{x} = \mathbf{Q}\mathbf{X} + \mathbf{c}, \tag{19}$$

where the mapping $\mathbf{Q}$ is proper-orthogonal.

7 Rotation Sequences

If $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are proper-orthogonal, then it follows that $\mathbf{Q}_1\mathbf{Q}_2$ is also proper-orthogonal. One of the first things we learn about rotation operators is that generally, $\mathbf{Q}_1\mathbf{Q}_2 \neq \mathbf{Q}_2\mathbf{Q}_1$.

Theorem: Rotation operators commute if and only if their axial vectors are parallel.
Proof: ??

Now we consider the orientation of a rigid body subject to a sequence of rotations. Let $\mathbf{u}_i$ be a set of n unit vectors, and let θ_i be a sequence of corresponding angles. We can consider the $\mathbf{u}_i$'s to be either fixed in space or fixed to the rigid body. In both cases, the final orientation of the body depends on the *order* of application of the rotations $\mathbf{Q}_i = \mathbf{Q}(\mathbf{u}_i, \theta_i)$. We choose an order, and relabel the rotations so that the final orientation $\mathbf{Q}$ of the rigid body is given by

$$\mathbf{Q} = \mathbf{Q}_1 \mathbf{Q}_2 \cdots \mathbf{Q}_n \tag{20}$$

This sequence of rotations can be visualized as illustrated in Figure 1. Having chosen an order for the rotations, $\mathbf{Q}$ becomes a function of the angles θ_i. We allow these to change with time so that $\mathbf{Q} = \mathbf{Q}(t)$. Note that $\dot{\mathbf{Q}} = \frac{\partial \mathbf{Q}}{\partial \theta_i} \dot{\theta}_i$, and that

Figure 1: Mechanical contraption illustrating the effect of successive rotations on a rigid body. Our interest is in the orientation of the body furthest from the base.

$$\frac{\partial \mathbf{Q}}{\partial \theta_i} = \mathbf{Q}_1 \cdots \mathbf{Q}_{i-1} \frac{\partial \mathbf{Q}_i}{\partial \theta_i} \mathbf{Q}_{i+1} \cdots \mathbf{Q}_n. \tag{21}$$

It follows that

$$\begin{aligned} \dot{\mathbf{Q}}\mathbf{Q}^T = {} & \dot{\theta}_1 \frac{\partial \mathbf{Q}_1}{\partial \theta_1} \\ & + \dot{\theta}_2 \mathbf{Q}_1 \frac{\partial \mathbf{Q}_2}{\partial \theta_2} \mathbf{Q}_1^T \\ & + \dot{\theta}_3 \mathbf{Q}_1 \mathbf{Q}_2 \frac{\partial \mathbf{Q}_2}{\partial \theta_2} \mathbf{Q}_2^T \mathbf{Q}_1^T \\ & \vdots \\ & + \dot{\theta}_3 \mathbf{Q}_1 \mathbf{Q}_2 \cdots \mathbf{Q}_{n-1} \frac{\partial \mathbf{Q}_n}{\partial \theta_n} \mathbf{Q}_{n-1}^T \cdots \mathbf{Q}_2^T \mathbf{Q}_1^T. \end{aligned} \tag{22}$$

If $\mathbf{P}$ is proper-orthogonal and $\mathbf{S}$ is skew-symmetric with axial vector $\mathbf{s}$, then $\mathbf{P}\mathbf{S}\mathbf{P}^T$ is skew-symmetric with axial vector $\mathbf{P}\mathbf{s}$. It follows from (15) that the axial vector associated with each skew-symmetric $\frac{\partial \mathbf{Q}_i}{\partial \theta_i} \mathbf{Q}_i^T$ is given by $\mathbf{u}_i$, and so the axial vector associated with (22) is given by

$$\boldsymbol{\omega} = \dot{\theta}_1\mathbf{v}_1 + \dot{\theta}_2\mathbf{v}_2 + \cdots + \dot{\theta}_n\mathbf{v}_n, \tag{23}$$

where $\mathbf{v}_1 = \mathbf{u}_1$, and $\mathbf{v}_k = \mathbf{Q}_1 \cdots \mathbf{Q}_{k-1}\mathbf{u}_k$ for $k = 2, \ldots, n$. Note that $\mathbf{v}_k$ is the image of $\mathbf{u}_k$ in the mechanism in Figure 1 after all angles have been set to their current values.

We note that space fixed $\mathbf{u}_i$'s with rotations $(\mathbf{u}_i, \theta_i)$ taken in the order $(i_1, i_2, \ldots, i_n)$ give a net rotation that is the same as body fixed $\mathbf{u}_i$'s with rotations taken in the opposit order $(i_n, i_{n-1}, \ldots, i_1)$.

Theorem: If the unit vectors $\mathbf{u}_i$ span E^3, then under point of view i. a set of angles exists for which the composit rotation operator equals *any* rotation operator.
Proof: ??

Note that the Euler representation has $\mathbf{u}_2 = \mathbf{Q}_1\tilde{\mathbf{u}}_2$ and $\mathbf{u}_3 = \mathbf{Q}_2\mathbf{Q}_1\tilde{\mathbf{u}}_3$, thus we've verified the usual construction using Euler angles, and we've identified the singularities that cause it to fail.

7.1 Rotation Factoring

Here discuss decomposing an arbitrary proper-orthogonal tensor into a sequence of rotations about different axes. In particular, decompose Q into the product of three Q's, each with a row and column all zeros except the diagonal...

7.2 Angular Velocity

Determine angular velocity for the Euler case. Note that the construction applies to the derivative of $\mathbf{Q}$ orienting any rigid body with a current configuration given by successive rotations about n body fixed axes. $\mathring{\alpha}$ and $\boldsymbol{\omega}$, and $\mathring{\mathbf{Q}}$ versus $\dot{\mathbf{Q}}$

8 Topological Observations

We have chosen not to identify rotations that differ by integer multiples of 2π. This is because it is sometimes necessary to keep track of how many rotations have occured. Every rotation $(\mathbf{u}, \theta)$ for which $\theta \neq 0$ therefor corresponds to a unique vector $\theta\mathbf{u} \in E^3$, with $\theta = 0$ corresponding to the zero vector. Our set of rotations is therefore a *vector space*! Once objects in this space are identified with one another, we move to various manifolds, e.g., the set of rotation operators corresponds to the π radius ball in E^3 with "antipodal" points identified. (Choosing these boundary points uniquely took an extra line of code in `blah.m`. This is SO^3 or equivalently the projective ball... (see Frankel).

9 A Construction Due to Rodriguez

Here we present a geometric construction on the surface of a sphere, that requires no equations at all. Consider a rigid body that undergoes rotation about some axis $\mathbf{p}$ through some angle θ, and then that undergoes rotation about an axis $\mathbf{q}$ through an angle ϕ. A delightful geometric construction by Rodriguez gives the resulting axis and angle of the net rotation.

- Extend the rigid body to a rigid sphere in space, the surface of which we will call S_1. (Give photo of a super-ball with something embedded inside).
- Enclose S_1 in a transparent rigid spherical shell, that we will call S_2. This shell will remain fixed in space, and we'll rotate S_1 within it.
- Let p and q be two points on S_2, and let l be the great circle passing through them. (Assume the interesting case is chosen, where p and q are not on opposite sides of S_2).

- copy l from the shell S_2 onto the inner rigid sphere S_1, and rotate S_1 through an angle θ about the vector from the sphere center to p (in the right hand sense). (Think of sticking a push-pin through $p \in S_2$ so that it reaches S_1, and keeps the point initially coincident there from moving). Now scribe the rotated copy of l onto S_2, and label it l'.
- Next, perform a similar rotation of the inner sphere through an angle ϕ about the vector from the sphere center to q, and let l'' denote the image of l under this rotation.

Use this construction to decompose a single rotation into the sequence of multiple rotations... Keeping track of angles with arbitrary magnitudes is meaningful for instance for a rigid body stuck in goo, (like a string being wound around a rod).

10 Quaternions

We define Quaternions as the 4-dimensional vector space over $\mathbb{R}$ of ordered quadruples (q_0, q_1, q_2, q_3), with a product given by

$$\begin{aligned}(q_0, q_1, q_2, q_3)(r_0, r_1, r_2, r_3) = (&q_0 r_0 - q_1 r_1 - q_2 r_2 - q_3 r_3, \dots \\ &q_0 r_1 + r_0 q_1 + q_2 r_3 - q_3 r_2, \dots \\ &q_0 r_2 + r_0 q_2 + q_3 r_1 - q_1 r_3, \dots \\ &q_0 r_3 + r_0 q_3 + q_1 r_2 - q_2 r_1) \end{aligned} \tag{24}$$

Letting $\mathbf{i}_k$ denote the quaternion (q_0, q_1, q_2, q_3) with all components zero except $q_k = 1$, we observe that

$$\mathbf{i}_1^2 = \mathbf{i}_2^2 = \mathbf{i}_3^2 = -\mathbf{i}_0,\ \mathbf{i}_1\mathbf{i}_2 = \mathbf{i}_3, \text{ and } \mathbf{i}_2\mathbf{i}_1 = -\mathbf{i}_3 \tag{25}$$

Writing (q_0, q_1, q_2, q_3) as $(q_0, \mathbf{q})$ where $\mathbf{q} = [q_1, q_2, q_3]^T$ allows the quaternion product to be expressed using the familiar dot and cross products

$$(u_0, \mathbf{u})(v_0, \mathbf{v}) = (u_0 v_0 - \mathbf{u} \bullet \mathbf{v}, u_0 \mathbf{v} + v_0 \mathbf{u} + \mathbf{u} \times \mathbf{v}) \tag{26}$$

Although this expression couches the quaternion product from (24) in familiar terms, it fails to provide us with any intuition for what's going on. This is especially true in light of the Argand plane for complex numbers, in which a product comparable to (24) boils down to a rotation with scaling. It turns out that a comparably basic, earthy, viewpoint exists for quaternion products, in which they correspond to successive rotations of a three dimensional rigid body. To see this, we note that any quaternion $(q_0, \mathbf{q})$ can be written as $a(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\mathbf{u})$, where $a \geq 0$, $\theta \in (-\pi, \pi]$, and $\mathbf{u}$ is a unit vector. This expression is rather like the polar form for complex numbers. Letting $\mathbf{a} = a\mathbf{u}$, we see that a quaternion is fully characterized by a vector $\mathbf{a}$ in three space, and an angle θ. Although we had a vector $\mathbf{q}$ and a real number q_0 before, the new vector angle pair allow for an elegant interpretation of the quaternion product. This is the same as how moving from the Cartesian representation $a + bi$ of a complex number (two real numbers, a and b) to polar form $re^{i\theta}$ (two real numbers r and θ) makes it easy to understand the product of complex numbers. The elegant interpretation of the quaternion product is as follows:

> The product $(\mathbf{a}, \theta_a)(\mathbf{b}, \theta_b)$ of two quaternions is found as follows
>
> - first, rotate a rigid body through an angle θ_b about the unit vector in the direction of $\mathbf{b}$.
> - next, rotate the body through an angle θ_a about the unit vector in the direction of $\mathbf{a}$.
> - the successive rotations are equivalent to a single rotation through an angle γ about the unit vector $\mathbf{u}$.
> - the product of the two quaternions is given by $(ab\mathbf{u}, \gamma)$, where $a = \|\mathbf{a}\|$ and where $b = \|\mathbf{b}\|$.

References

[1] Sheldon Axler, *Linear Algebra Done Right*

[2] Kuipers, J., *Quaternions, Rotations, and blah.*